Dynamic Fair Division with Partial Information*

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Abstract

We consider the fundamental problem of fairly and efficiently allocating T indivisible items among n agents with additive preferences. The items become available over a sequence of rounds, and every item must be allocated immediately and irrevocably before the next one arrives. Previous work shows that when the agents' valuations for the items are drawn from known distributions, it is possible (under mild technical assumptions) to find allocations that are envy-free with high probability and Pareto efficient ex-post.

We study a *partial-information* setting, where it is possible to elicit ordinal but not cardinal information. When a new item arrives, the algorithm can query each agent for the relative rank of this item with respect to a subset of the past items. When values are drawn from i.i.d. distributions, we give an algorithm that is envy-free and $(1-\epsilon)$-welfare-maximizing with high probability. We provide similar guarantees (envy-freeness and a constant approximation to welfare with high probability) even with minimally expressive queries that ask for a comparison to a single previous item. For independent but non-identical agents, we obtain envy-freeness and a constant approximation to Pareto efficiency with high probability. We prove that all our results are asymptotically tight.

1 Introduction

We consider the following fundamental problem in fair division. A set of T indivisible items, arriving one at a time, must be allocated among a set of n agents with additive preferences. The value $v_{i,t}$ that agent i has for the item in round t is realized once the item arrives. Each item must be allocated immediately and irrevocably upon arrival, and we ask that the overall allocation is *fair* and *efficient*.

Previous work on this problem shows that, despite the uncertainty about future valuations, one can achieve simultaneous fairness and efficiency when agents' values are stochastic. Specifically, when each $v_{i,t}$ is drawn i.i.d. from a distribution D, the simple algorithm that maximizes welfare — each item is allocated to the agent with the highest value — is envy-free with high probability and (obviously) ex-post Pareto efficient [DGK+14, KPW16]. The same guarantee holds for independent and non-identical agents ($v_{i,t}$ is drawn from an agent-specific distribution D_i) for the algorithm that maximizes weighted welfare [BG22]. Even when agents' valuations for an item are correlated (but items are independent), Pareto efficiency ex-post is compatible with strong fairness guarantees ("envy-freeness with high probability or envy-freeness up-to-one item ex-post") [ZP20].

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Despite the computational simplicity of (most of) the aforementioned algorithms, an unappealing aspect, especially from a practical perspective, is the requirement that agents report an exact numerical value for each item. Eliciting expressive additive valuations can be impractical, e.g., due to agents' cognitive limitations. Motivated by such considerations, a growing body of work in AI studies what can be achieved by algorithms that only elicit *ordinal* information. This idea originates from [PR06], who defined the notion of distortion to measure the worst-case deterioration of an aggregate cardinal objective (e.g., utilitarian social welfare) due to only having access to preferences of limited expressiveness, particularly ordinal rankings. Recent works prove bounds on the distortion in the context of many problems in social choice, e.g., voting [CNPS17, GKM17, MSW20, MW19, Kem20a, Kem20b, GHS20], matching [FRFZ14], and participatory budgeting [BNPS21]; see [AFRSV21] for a recent survey.

In this paper, we study the power and limits of eliciting ordinal information in dynamic fair division. The value $v_{i,t}$ of agent i for item t is drawn from an *unknown* distribution upon arrival, and the algorithm is provided, from each agent, partial ordinal information about this item, e.g., its rank relative to the past items allocated to this agent, or even just a single past item allocated to this agent. Under what distributional assumptions and elicitation constraints, can we simultaneously achieve qualitative fairness and efficiency? We answer these questions.

1.1 Our Contribution

We start by establishing a separation between the cardinal setting and our ordinal one. Pareto efficiency alone is trivial (allocate all goods to the same agent) and, in the cardinal setting, it is known that Pareto efficiency ex-post is compatible with envy-freeness with high probability (as long as agents are independent). We prove (Theorem 4) that in our setting, even for the case of two i.i.d. agents and any *known* distribution, envy-freeness with high probability is incompatible with even a very mild notion of (exact) Pareto efficiency, one-swap-Pareto efficiency, which requires that there is no beneficial one-to-one trade of items between agents (but allows for improvements via many-to-many trades of items).

We proceed to give an essentially matching positive result. For any number of i.i.d. agents and an unknown distribution D, there exists an algorithm (Algorithm 1) that is envy-free with high probability and guarantees a $(1-\varepsilon)$-approximation to the optimal utilitarian social welfare (the sum of utilities), for all $\varepsilon > 0$, with high probability (Theorem 5). When an item arrives, the algorithm learns for each agent i its relative rank compared to a subset of prior items allocated to agent i (but otherwise knows nothing about the underlying numerical valuation). Our algorithm works in epochs. Each epoch has an exploration/sampling phase, where each agent gets a pre-determined number of items, followed by an exploitation/ranking phase, where each fresh item is given to an agent whose empirical quantile is largest. The goal is to make a sublinear number of errors compared to the "ideal" algorithm that allocates each item to the agent with the highest true quantile. The algorithm has to balance the need for sampling, which leads to more accurate empirical quantiles, against the number of inefficient allocations made while sampling. A significant technical barrier is that we cannot fix a target accuracy because the underlying distribution is unknown. That is, for every fixed accuracy for the empirical quantiles, there exists a distribution for which this accuracy is not good enough for even a constant approximation to the optimal welfare. Instead, we need to make our epochs progressively longer, thereby guaranteeing progressively better bounds.

Given this strong positive result, we explore the limits of what we can achieve when further restricting the amount of information available. What if each agent can remember only a *single* item previously allocated to them, and the fresh item is compared to just this one item?[1] Surprisingly, the aforementioned positive result can almost be recovered even in this very restrictive setting. We prove that there exists an algorithm (Algorithm 2) that is envy-free with high probability and guarantees a $(1-1/e)^2-\varepsilon$ approximation to the optimal welfare with high probability, for all $\varepsilon > 0$ (Theorem 9). Our algorithm again proceeds in epochs with gradually increasing exploration and exploitation phases; this time the goal is a sublinear number of differences compared to allocating each item to a uniformly random agent with quantile at least $1-1/n$, which we prove is envy-free and approximately efficient (Lemmas 2 and 1). When exploring, the algorithm puts a new item in memory, estimates its quantile, and rejects it if not sufficiently close to $1-1/n$. We need to sample enough to ensure high confidence in the estimated quantile, but also account for the additional sampling since an item's quantile value might be far from $1-1/n$ to begin with; several technical details need to be accounted for. We give a near-matching lower bound: no algorithm with a memory of one item can achieve a 0.999−approximation to the social welfare with high probability; therefore a constant approximation (which Algorithm 2 provides) is all we can hope for.

Finally, we relax the i.i.d. condition and study agents that are independent but not identical; each agent i's values are drawn from an unknown distribution D_i. Even with unbounded memory, we show that it is impossible to get a $\frac{1+\sqrt{5}}{4} \approx .809$ approximation to Pareto efficiency with probability more than $2/3$, even for two agents (Theorem 12). At the same time, we prove that Algorithms 1 and 2 are envy-free and $1/e$ approximately Pareto efficient with high probability! Note that, though both algorithms give the same formal guarantees and Algorithm 2 elicits strictly less information, one might still prefer to use Algorithm 1 since it has significantly shorter exploration phases.

We leave the study of correlated agents as an interesting open problem. Finally, we note that beyond stochastic valuations, [BKPP18] show that it is possible to achieve sublinear envy by randomly allocating every item when agents' valuations are adversarially generated (and this is optimal); however, sublinear envy is incompatible with non-trivial efficiency even in the cardinal setting [ZP20].

1.2 Related Work

A number of works study fair division under ordinal preferences, e.g., [AGMW15, BEL10, BBL+17, NNR17], but often these models do not assume an underlying cardinal model and work directly on the ordinal preferences. [ABM16] assume underlying cardinal information and, among other results, bound the approximation ratio of truthful mechanisms that elicit rankings. Closer to our work, [HS21] study rules that have access to the ranking of the top-k items of each agent and bound the ratio of the social welfare of the allocation returned by a rule in the worst case. They also characterize the value of k needed to achieve prominent notions of fairness, namely envy-freeness up to one item (EF1) and approximate maximin share guarantee (MMS), as well as bound the loss in efficiency incurred due to fairness constraints in this setting.

Our work contributes to the growing literature in dynamic fair division [KPS14, AAGW15, FPV15, FPV17, BKPP18, HPPZ19, ZP20, GPT21, BKM22, GBI21, VPF21] (and we note that the welfare-maximizing algorithms of [DGK+14, KPW16, BG22] work in the dynamic setting, even though the their settings are not explicitly dynamic). To the best of our knowledge, we are the first to study imperfect expressivity in a dynamic setting in fair division.

[1] So, the algorithm only learns if the new item is better or worse than the item in memory and may, at that time, choose to replace the item in memory.

2 Preliminaries

A set of T indivisible items/goods, labeled by $\mathcal{G} = \{1, 2, \cdots, T\}$, needs to be allocated to a set of n agents, labeled by $\mathcal{N} = \{1, \ldots, n\}$. Agent $i \in \mathcal{N}$ assigns a value $v_{i,t} \in [0, 1]$ to item $t \in \mathcal{G}$. We assume agents have *additive* valuation functions, so $v_i(S) = \sum_{t \in S} v_{i,t}$ for $S \subseteq \mathcal{G}$. An allocation A is a partition of the items into bundles $A_1, \ldots, A_n$, where A_i is the set of items assigned to agent $i \in \mathcal{N}$. Each allocation has an associated utility profile $v(A) = (v_1(A_1), \ldots, v_n(A_n))$.

Items arrive online, one per round. The agents' valuations for the item in round t (the t-th item) are realized when the item arrives. Every item is allocated immediately and irrevocably before moving on to the next round. We write $\mathcal{G}^t = \{1, 2, \cdots, t\}$ for the set of items that arrived in the first t rounds, and A_i^t for the allocation of agent i after the t-th item was allocated. We consider two different models which specify how values are generated. In the **i.i.d. model**, the value of agent i for item t is independently drawn from an *unknown* distribution D with CDF F, i.e., $v_{i,t} \sim D$ for all $i \in \mathcal{N}$ and $t \in \mathcal{G}$. In the **non-i.i.d. model**, the value of item t for agent i is independently drawn from an *unknown*, agent-dependent distribution D_i with CDF F_i, i.e., $v_{i,t} \sim D_i$ for all $i \in \mathcal{N}$ and $t \in \mathcal{G}$. We write X_i for the random variable for i's valuation, and $X_{i,t}$ for the random variable for i's valuation for item t. It is often convenient to work directly with the quantile of an agent's value rather than the value itself; let $Q_i = F_i(X_i)$ and $Q_{i,t} = F_i(X_{i,t})$ respectively be the random variable denoting the quantile of agent i the associated item. Note that all Q_i and $Q_{i,t}$ are i.i.d. and follow a Unif$[0, 1]$ distribution. Unless explicitly stated otherwise, we assume all distributions are continuous (i.e., do not have point masses).

Ordinal Information. We assume the realizations $v_{i,t}$ are not available. Instead, our algorithms have access to *ordinal* information. Specifically, given current item t, the algorithm can access each agent's *ranking* for $S = \{t\} \cup M$, $M \subseteq \mathcal{G}^{t-1}$. The size of M, which we will informally refer to as the *memory size*, determines the complexity of eliciting information from each agent. In one extreme, agent i compares a new item t to a single item they had previously received, i.e., $M \subseteq A_i^{t-1}$, $|M| \leq 1$. In the other extreme, t is compared to all previous items she received, so $M = A_i^{t-1}$. We write $\sigma_i(S)$ for the ranking of agent i for a subset S of the items, and $\sigma_i^{-1}(S, t)$ for the position of item $t \in S$ with respect to a subset S according to agent i. The highest value item is in position 1 and the lowest in position $|S|$. For example, if $S = \{1, 4\}$, $v_{i,1} = 1$ and $v_{i,4} = 0.1$, $\sigma_i(S) = (1 \succ 4)$, $\sigma_i^{-1}(S, 1) = 1$ and $\sigma_i^{-1}(S, 4) = 2$.

Algorithms. An algorithm $\mathcal{A}$, in each step t, queries each agent for ordinal information with respect to some subset M and then makes a (possibly randomized) allocation decision based on this ordinal information and the history so far. An instance of our problem is parameterized by the number of agents n and the (unknown) value distributions $D_1, \ldots, D_n$. Let $\mathcal{E}_P(t)$ be the event that some algorithm satisfies property P (e.g., envy-freeness or PO or ε-welfare) at time t. We are interested in the probability that an algorithm satisfies certain properties in the limit, as the number of rounds tends to infinity, where the randomness is over the random choices of the algorithm as well as the randomness in the valuations.

Definition 1. An algorithm satisfies P with high probability if $\lim_{t \to \infty} \Pr[\mathcal{E}_P(t)] = 1$.

Note that this definition of high probability allows for dependency on n and the underlying distributions (i.e., they are treated as constants).

Efficiency notions. An allocation A Pareto dominates an allocation A', denoted $A \succ A'$, when $v_i(A_i) \geq v_i(A'_i)$ for all $i \in \mathcal{N}$ and there exists $j \in \mathcal{N}$ with $v_j(A_j) > v_j(A'_j)$. An allocation A is *Pareto efficient* or *Pareto optimal* (PO) if there is no feasible (integral) allocation that Pareto dominates it. An allocation A' is in the (one) swap-neighborhood of A when it can be created from A with a single exchange of items between one pair of agents. Formally, there exist $i, j \in \mathcal{N}$ and items $z_j \in A_j$ and $z_i \in A_i$ so that $A'_i = (A_i \setminus \{z_i\}) \cup \{z_j\}$, $A'_j = (A_j \setminus \{z_j\}) \cup \{z_i\}$, and $A'_k = A_k$ for all other agents $k \neq i, j$. An allocation A is *one-swap Pareto optimal* (SPO) if it is undominated by any allocation in its swap-neighborhood. We use a notion of approximate efficiency defined by [RF90] according to which an allocation A is *α-Pareto efficient* when $v(A)/\alpha$ is undominated.

The social welfare of an allocation A is $\mathsf{sw}(A) = \sum_{i \in \mathcal{N}} v_i(A_i)$. Let allocation A^* denote a (social) welfare optimal allocation for which $\mathsf{sw}(A^*) \geq \mathsf{sw}(A)$ for all feasible allocations A. An allocation provides an α-approximation to welfare if $\mathsf{sw}(A) \geq \alpha \cdot \mathsf{sw}(A^*)$.

Fairness notions. We focus on a prominent notion of fairness called *envy-freeness*. An allocation $A^T = (A^T_1, \ldots, A^T_n)$ of T items is *envy-free* (EF) when $v_i(A^T_i) \geq v_i(A^T_j)$ for all $i, j \in \mathcal{N}$, and c-strongly-envy-free (c-strong-EF) when $v_i(A_i) \geq v_i(V_j) + cT$.

3 Ideal Quantile-based Algorithms.

For our analysis, it will be useful to compare our algorithms with ideal algorithms that know *exact* quantile values for every item (and, in fact, several of our lower bounds apply to these stronger algorithms too). Two ideal algorithms of interest are (1) quantile maximization, which allocates each item to the agent with the highest quantile value for it, and (2) "$q-$threshold," which allocates each item uniformly at random among agents whose quantile is at least q (and uniformly at random over all agents, if all quantile values are less than q).

In the i.i.d. model, quantile maximization is the same as value maximization, and thus envy-free with high probability and ex-post welfare optimal. The property we will use is c-strong envy-freeness, for some distribution-dependent constant c, which we state as Lemma 1. This was essentially proved by [DGK+14]; we provide an alternate proof that also works, essentially unchanged, for the $\frac{n-1}{n}$-threshold algorithm; it can be found in Appendix A.1.

Lemma 1. *[Essentially [DGK+14].] In the i.i.d. and non-i.i.d. models, the quantile maximization algorithm and the $\frac{n-1}{n}$-threshold algorithm are c-strongly-envy-free, with high probability, where the constant $c = \min_{i \in \mathcal{N}} (\mathbb{E}[X_i \mid Q_i \geq 1/2] - \mathbb{E}[X_i])/(4n)$.*

Note that c is strictly positive since our distributions are continuous. In the i.i.d. model, we show that the $\frac{n-1}{n}$-threshold algorithm gives a $\left(1 - \frac{1}{e}\right)^2 - \varepsilon$ approximation to welfare (Lemma 2) with high probability.

Lemma 2. *In the i.i.d. model, the $\frac{n-1}{n}$-threshold algorithm guarantees a $\left(\left(1 - \frac{1}{e}\right)^2 - \varepsilon\right)$-approximation to welfare, with high probability, for all $\varepsilon > 0$.*

Proof. Let F be the CDF of an arbitrary continuous distribution. The expected contribution of an item to the welfare of the threshold algorithm is at least

$$\mathop{\mathbb{E}}_{Q \sim \text{Unif}[0,1]} \left[F^{-1}(Q) \mid Q \geq \frac{n-1}{n} \right] \cdot \Pr_{\vec{Q} \sim \text{Unif}[0,1]^n} \left[\max_{i \in \mathcal{N}} Q_i \geq \frac{n-1}{n} \right].$$

For the first term we have

$$
\begin{aligned}
\mathop{\mathbb{E}}_{Q\sim\mathrm{Unif}[0,1]}\left[F^{-1}(Q) \mid Q \geq \frac{n-1}{n}\right] &= \mathop{\mathbb{E}}_{Q\sim\mathrm{Unif}[\frac{n-1}{n},1]}\left[F^{-1}(Q)\right] \\
&= \left(\int_{\frac{n-1}{n}}^{1} F^{-1}(q)\cdot f_{\mathrm{Unif}[\frac{n-1}{n},1]}(q)\,\mathrm{d}q\right) \\
&= \left(\int_{\frac{n-1}{n}}^{1} F^{-1}(q)\cdot n\,\mathrm{d}q\right) \\
&\geq^{(f_{\mathrm{Beta}[n,1]}(x)=nx^{n-1})} \left(\int_{\frac{n-1}{n}}^{1} F^{-1}(q)\cdot f_{\mathrm{Beta}[n,1]}(q)\,\mathrm{d}q\right) \\
&= \mathop{\mathbb{E}}_{Q\sim\mathrm{Beta}[n,1]}\left[F^{-1}(Q) \mid Q \geq \frac{n-1}{n}\right]\cdot \mathop{\Pr}_{Q\sim\mathrm{Beta}[n,1]}\left[Q \geq \frac{n-1}{n}\right] \\
&\geq \mathop{\mathbb{E}}_{Q\sim\mathrm{Beta}[n,1]}\left[F^{-1}(Q)\right]\cdot \mathop{\Pr}_{Q\sim\mathrm{Beta}[n,1]}\left[Q \geq \frac{n-1}{n}\right] \\
&= \mathop{\mathbb{E}}_{\vec{Q}\sim\mathrm{Unif}[0,1]^n}\left[F^{-1}(\max_{i\in\mathcal{N}} Q_i)\right]\cdot \mathop{\Pr}_{\vec{Q}\sim\mathrm{Unif}[0,1]^n}\left[\max_{i\in\mathcal{N}} Q_i \geq \frac{n-1}{n}\right],
\end{aligned}
$$

where we used the fact that the maximum of n draws from $U[0,1]$ follows a $\mathrm{Beta}(n,1)$. The expected contribution to the welfare is thus at least

$$
\mathop{\mathbb{E}}_{\vec{Q}\sim\mathrm{Unif}[0,1]^n}\left[F^{-1}(\max_{i\in\mathcal{N}} Q_i)\right]\left(1-\left(1-\frac{1}{n}\right)^n\right)^2 \geq \left(1-\frac{1}{e}\right)^2 \mathop{\mathbb{E}}_{\vec{Q}\sim\mathrm{Unif}[0,1]^n}\left[F^{-1}(\max_{i\in\mathcal{N}} Q_i)\right]
$$

Finally, for any fixed $\varepsilon > 0$, standard Chernoff bounds tell us that with high probability, the optimal welfare of T items is at most $T\cdot(1+\varepsilon/2)\,\mathbb{E}_{\vec{Q}\sim\mathrm{Unif}[0,1]^n}\left[F^{-1}(\max_{i\in\mathcal{N}} Q_i)\right]$ while the welfare of the threshold algorithm is at least $T\cdot(1-\varepsilon/2)\left(1-\frac{1}{e}\right)^2 [\mathbb{E}_{\vec{Q}\sim\mathrm{Unif}[0,1]^n}\left[F^{-1}(\max_{i\in\mathcal{N}} Q_i)\right]$. Indeed, the expected optimal welfare is equal to $T\cdot\mathbb{E}_{\vec{Q}\sim\mathrm{Unif}[0,1]^n}\left[F^{-1}(\max_{i\in\mathcal{N}} Q_i)\right]$, the sum of T i.i.d. random variables with expectation $\mathbb{E}_{\vec{Q}\sim\mathrm{Unif}[0,1]^n}\left[F^{-1}(\max_{i\in\mathcal{N}} Q_i)\right]$. The standard multiplicative Chernoff bound says that the sum of i.i.d. variables exceeds $(1+\epsilon/2)$ times its expectation μ is at most $\exp\left(-\mu\epsilon^2/12\right)$. Plugging in $\mu = T\cdot\mathbb{E}_{\vec{Q}\sim\mathrm{Unif}[0,1]^n}\left[F^{-1}(\max_{i\in\mathcal{N}} Q_i)\right]$, we get the desired statement. The statement about the welfare of the threshold algorithm follows similarly. Thus, the algorithm is a

$$
\left(1-\frac{1}{e}\right)^2\cdot(1-\varepsilon/2)/(1+\varepsilon/2) \geq \left(1-\frac{1}{e}\right)^2(1-\varepsilon) \geq \left(1-\frac{1}{e}\right)^2 - \varepsilon
$$

approximation to welfare, with high probability. □

Finally, we prove that both ideal algorithms are approximately efficient. Let $\mathcal{P}^*$ be the following property of an allocation: all items such that exactly one agent has quantile values at least $1-1/n$ are in the bundle of this agent. Both ideal algorithms (quantile maximization and $1-1/n$-threshold) satisfy $\mathcal{P}^*$. We prove that, in the non-i.i.d. model, $\mathcal{P}^*$ implies an almost $1/e$ approximation to efficiency. Our proof uses the fact that there is a (roughly) $1/e$ probability that exactly one agent has the high quantile, so the value of an agent's bundle in an algorithm that satisfies $\mathcal{P}^*$ is, with high probability, a $1/e$ approximation to their value for their T/n most valuable items. Therefore, when considering an alternate allocation A', the agent in A' that gets at most T/n items cannot be improved upon by more than a $1/e$ factor.

Lemma 3. *In the non-i.i.d. model, every algorithm whose allocations satisfy $\mathcal{P}^*$ is $(1/e-\varepsilon)$-Pareto optimal, with high probability, for all $\varepsilon > 0$.*

Proof. Fix an $\varepsilon \in (0,1)$, and choose ε' such that $\frac{1-\varepsilon'}{(1+\varepsilon')^2} \cdot \frac{1}{e} > \frac{1}{e} - \varepsilon$ (using $\varepsilon' = \varepsilon/3$ will do). Fix distributions with CDFs $F_1, \dots, F_n$ for each agent $i \in \mathcal{N}$, and a time T. Suppressing the superscript, for ease of notation, let $A_i = A_i^T$ be the bundle allocated at time T to each agent i by an algorithm that satisfies $\mathcal{P}^*$. Let A_i^{top} be the set of the T/n most valuable items for each agent i. Let $A_i^{\text{high}} = \{\, t \in \mathcal{G}^T \;\mid\; F_i(v_{i,t}) \geq 1 - \frac{1+\varepsilon'}{n} \,\}$ be the set of items that agent i has "high" value for, in the sense that they come from the top $\frac{1+\varepsilon'}{n}$ portion of their distribution. We show the following $3n$ events, $\mathcal{E}_{ij}$ for $i \in \mathcal{N}$ and $j \in \{1,2,3\}$, occur simultaneously with high probability (in T).

1. $\mathcal{E}_{i1}$: $v_i(A_i^{\text{top}}) \leq v_i(A_i^{\text{high}})$.
2. $\mathcal{E}_{i2}$: $v_i(A_i^{\text{high}}) \leq T \cdot \frac{(1+\varepsilon')^2}{n} \mathbb{E}_{Q\sim \text{Unif}[1-1/n,1]}[F^{-1}(Q)]$.
3. $\mathcal{E}_{i3}$: $v_i(A_i) \geq T \cdot \frac{1-\varepsilon'}{en} \mathbb{E}_{Q\sim \text{Unif}[1-1/n,1]}[F^{-1}(Q)]$.

Each of these individually will follow from a straightforward application of Hoeffding's inequality or Chernoff bounds, showing they each individually occur with probability exponentially close to 1 in T. This implies that they all occur simultaneously with high probability. Finally, we will show that conditioned on all $3n$ occurring, the allocation is $(1/e-\varepsilon)$-PO.

Let us begin with $\mathcal{E}_{i1}$ for each agent i. The event occurs when there are at least T/n items $t \in \mathcal{G}^T$ such that $F_i(v_{i,t}) \geq 1 - \frac{1+\varepsilon'}{n}$. Each item independently satisfies this property ($F_i(v_{i,t}) \geq 1 - \frac{1+\varepsilon'}{n}$) with probability $\frac{1+\varepsilon'}{n}$. Hence the probability this does not occur is at most $2\exp\left(-2\varepsilon'^2 T\right)$.

Next, consider $\mathcal{E}_{i2}$ for each agent i. The expected contribution of each item to $v_i(A_i^{\text{high}})$ is

$$\mathop{\mathbb{E}}_{Q\sim \text{Unif}[0,1]}\left[F_i^{-1}(Q) \cdot \mathbb{I}\left[Q \geq 1 - \frac{1+\varepsilon'}{n}\right]\right] = \frac{1+\varepsilon'}{n} \mathop{\mathbb{E}}_{Q\sim \text{Unif}[1-\frac{1+\varepsilon'}{n},1]}[F_i^{-1}(Q)] \leq \frac{1+\varepsilon'}{n} \mathop{\mathbb{E}}_{Q\sim \text{Unif}[1-\frac{1}{n},1]}[F_i^{-1}(Q)].$$

We now use the following multiplicative version of the Chernoff bound,

$$\Pr\left[\sum_i X_i \geq (1+\delta)\sum_i \mathbb{E}[X_i]\right] \leq \exp\left(-\frac{\delta^2}{3}\sum_i \mathbb{E}[X_i]]\right),$$

to conclude that the probability that $v_i(A_i^{\text{high}})$ exceeds $T \cdot \frac{(1+\varepsilon')^2}{n} \mathbb{E}_{Q\sim \text{Unif}[1-1/n,1]}[F^{-1}(Q)] \geq (1+\varepsilon') \cdot \mathbb{E}[v_i(A_i^{\text{high}})]$ is at most $\exp\left(-\frac{\varepsilon'^2(1+\varepsilon')\mathbb{E}_{Q\sim \text{Unif}[1-\frac{1}{n},1]}[F_i^{-1}(Q)]}{3n} \cdot T\right)$.

Finally, consider $\mathcal{E}_{i3}$ for each agent i. We will show that the expected contribution of each item to $v_i(A_i)$ is at least $\frac{1}{en} \cdot \mathbb{E}_{Q\sim \text{Unif}[1-\frac{1}{n},1]}[F_i^{-1}(Q)]$. Indeed, consider an item such that the quantile for agent i is $Q_i > 1 - 1/n$ while $Q_j < 1 - 1/n$ for all agents $j \neq i$. This occurs with probability $\frac{1}{n} \cdot \left(1 - \frac{1}{n}\right)^{n-1} \geq \frac{1}{en}$, and when this occurs, since the algorithm satisfies $\mathcal{P}^*$, it must allocate the item to i. Further, when this does occur, the expected value of such an item is $\mathbb{E}_{Q\sim\text{Unif}[1-\frac{1}{n},1]}[F_i^{-1}(Q)]$, since it is independent of the other agent's values. Hence the expectation is at least $\frac{1}{en}\mathbb{E}_{Q\sim\text{Unif}[1-\frac{1}{n},1]}[F_i^{-1}(Q)]$. Finally, we again use a multiplicative Chernoff bound to show that

$$\Pr\left[v_i(A_i) \le (1-\epsilon')\cdot\frac{T}{en}\mathop{\mathbb{E}}_{Q\sim\text{Unif}[1-\frac{1}{n},1]}[F_i^{-1}(Q)]\right] \le \exp\left(-\frac{\varepsilon'^2\,\mathbb{E}_{Q\sim\text{Unif}[1-\frac{1}{n},1]}[F_i^{-1}(Q)]}{2en}\cdot T\right).$$

Now, suppose that $\mathcal{E}_{ij}$ hold for all $i \in \mathcal{N}$ and $j \in \{1,2,3\}$. We show that this implies the allocation $A_1,\dots,A_n$ is $(1/e-\varepsilon)$-PO. Fix an arbitrary allocation $A'_1,\dots,A'_n$. We show there exists an agent $i \in \mathcal{N}$ such that $v_i(A'_i) < \frac{v_i(A_i)}{1/e-\varepsilon}$. First, there must be some agent i such that $|A'_i| \le T/n$. Since A'_i can be at most as valuable as the most-valuable T/n items, we have

$$\begin{aligned} v_i(A'_i) &\le v_i(A_i^{\text{top}}) \\ &\le^{(\mathcal{E}_{i1})} v_i(A_i^{\text{high}}) \\ &\le^{(\mathcal{E}_{i2})} T\cdot\frac{(1+\varepsilon')^2}{n}\mathop{\mathbb{E}}_{Q\sim\text{Unif}[1-1/n,1]}[F^{-1}(Q)] \\ &\le^{(\mathcal{E}_{i3})} \cdot\frac{(1+\varepsilon')^2}{(1-\varepsilon')(1/e)}v_i(A_i) \\ &< \frac{1}{1/e-\varepsilon}v_i(A_i), \end{aligned}$$

as needed. □

4 Unbounded memory in the i.i.d. model

We explore some fundamental limits of our setting. Efficiency by itself is easy: allocate all items to the same agent. However, in contrast to the cardinal setting, we find one-swap Pareto efficiency is incompatible with envy-freeness with high probability, even for two i.i.d. agents, and even when the underlying distribution is *known*.

Theorem 4. *In the i.i.d. model, even for $n = 2$ agents, there does not exist an algorithm $\mathcal{A}$ which is one-swap Pareto efficient and envy-free with high probability, even when values are sampled according to D, for any continuous, bounded and known value distribution D.*

Proof. Fix an arbitrary, continuous value distribution D and an algorithm $\mathcal{A}$.

As the agents are a priori identical, we can assume without loss of generality that $\mathcal{A}$ gives the first item to agent 1. We will show that, with a positive probability, this decision becomes an irrevocable "mistake," in the sense that agent 2 really liked the item and agent 1 did not. This mistake will make envy-freeness and one-swap PO incompatible.

First, we find values to make this mistake sufficiently bad. Let $g : [0,1] \to [0,1]$ be the function $g(q) = \mathbb{E}[X \mid X \le F^{-1}(q)]/\mathbb{E}[X]$, which maps a quantile q to the ratio of the expected value of an item below quantile q to the expected value of an arbitrary item. g is a continuous increasing function with $g(1) = 1$, so there is some quantile $\hat{q} < 1$ such that $g(\hat{q}) \ge 0.9$. Let $q_2^* = \max(\hat{q}, 0.9)$. Since g is increasing, $g(q_2^*) \ge g(\hat{q}) \ge 0.9$. Let $q_1^* = 0.1$, $v_1^* = F^{-1}(q_1^*)$ and $v_2^* = F^{-1}(q_2^*)$. Let

$\mathcal{E}^{\text{mistake}}$ be the event that $X_{1,1} < v_1^*$ and $X_{2,1} > v_2^*$. Define $c := \Pr[\mathcal{E}^{\text{mistake}}] = (1 - q_2^*) \cdot q_1^*$ to be the probability that $\mathcal{E}^{\text{mistake}}$ occurs. D is continuous, so $c > 0$. Our lower bound on the probability that the allocation at step t violates either envy-freeness or one-swap PO will only depend on c.

Let $\mathcal{E}_j$ be the event that for item j we have that both $X_{1,j} \geq v_1^*$ and $X_{2,j} \leq v_2^*$. If $\mathcal{E}^{\text{mistake}}$ occurs, the only way to maintain one-swap Pareto efficiency is to allocate item j to agent 1 every time $\mathcal{E}_j$ occurs; otherwise, swapping items 1 and j between the two agents yields a Pareto improvement. This constraint will make envy-freeness unlikely.

Let $\mathcal{E}^{\text{manyhigh}}(t)$ be the event $\sum_{j=2}^{t} X_{2,j} \cdot \mathbb{I}[\mathcal{E}_j] \geq (t-1) \cdot 0.7 \cdot \mathbb{E}[X]$. In other words, $\mathcal{E}^{\text{manyhigh}}(t)$ occurs when agent 2 has a high value for items j, $2 \leq j \leq t$, for which $\mathcal{E}_j$ occurs (i.e., the items that must be given to agent 1 in order to satisfy one-swap PO). Let $\mathcal{E}^{\text{normalval}}(t)$ denote the event that $\sum_{j=2}^{t} X_{2,j} \leq (t-1) \cdot 1.1 \cdot \mathbb{E}[X]$. We first show that for sufficiently large t, the probability that both $\mathcal{E}^{\text{manyhigh}}(t)$ and $\mathcal{E}^{\text{normalval}}(t)$ occur is at least $1/2$. To do so, we prove each event occurs with probability at least $3/4$, and then apply a union bound.

First, since each $X_{1,j}$ and $X_{2,j}$ are independent, $\Pr[\mathcal{E}_j] \geq 0.9 \cdot 0.9 = 0.81$, and $\mathbb{E}[X_{2,j}|\mathcal{E}_j] = \mathbb{E}[X_{2,j} \mid X_{2,j} \leq v_2^*]$. Also, from the definition of $g(\hat{q})$ and the choice of q_2^*, $\mathbb{E}[X_{2,j} \mid X_{2,j} \leq v_2^*] \geq 0.9 \cdot \mathbb{E}[X]$. It follows that $\mathbb{E}[X_{2,j} \cdot \mathbb{I}[\mathcal{E}_j]] = \mathbb{E}[X_{2,j}|\mathcal{E}_j] \cdot Pr[\mathcal{E}_j] \geq 0.729 \cdot \mathbb{E}[X]$. A straightforward Chernoff bound establishes that $\Pr[\mathcal{E}^{\text{manyhigh}}(t)] \geq 3/4$ for t at least $\frac{6}{\mathbb{E}[X]}$.

Let $Y_j = X_{2,j} \cdot \mathbb{I}[\mathcal{E}_j]$ for all j. Then, $\mathbb{E}[Y_j] \geq 0.729 \cdot \mathbb{E}[X]$, and $\mathbb{E}[\sum_{j=2}^{T} Y_j] \geq (t-1) \cdot 0.729 \cdot \mathbb{E}[X]$. We are interested in the probability that $\sum_{j=2}^{t} Y_j$ is at least $(t-1) \cdot 0.7 \cdot \mathbb{E}[X]$, i.e., the probability that $\sum_{j=2}^{t} Y_j$ is at least $\frac{0.7}{0.729}$ its expectation.

We use the following Chernoff bound: Let $Y_1, \ldots, Y_n$ be independent random variables that take values in $[0,1]$, and let Y be their sum. Then, for all $\delta \in [0,1)$, $\Pr[Y \leq (1-\delta)\,\mathbb{E}[Y]] \leq e^{-\frac{\mathbb{E}[Y]\delta^2}{2}}$.

Continuing our derivation:

$$\begin{aligned}
\Pr\left[\sum_{j=2}^{t} Y_j \geq (t-1) \cdot 0.7 \cdot \mathbb{E}[X]\right] &= \Pr\left[\sum_{j=2}^{t} Y_j \geq \frac{0.7}{0.79}\,\mathbb{E}[\sum_{j=2}^{t} Y_j]\right] \\
&= 1 - \Pr\left[\sum_{j=2}^{t} Y_j < \frac{0.7}{0.79}\,\mathbb{E}[\sum_{j=2}^{t} Y_j]\right] \\
&\geq 1 - \Pr\left[\sum_{j=2}^{t} Y_j \leq 0.89\,\mathbb{E}[\sum_{j=2}^{t} Y_j]\right] \\
&\geq 1 - \exp\left(-\frac{\mathbb{E}[\sum_{j=2}^{t} Y_j](0.89)^2}{2}\right),
\end{aligned}$$

which is at least $3/4$ when $\frac{\mathbb{E}[\sum_{j=2}^{t} Y_j](0.89)^2}{2}$ is at least $\ln(4)$, or, equivalently, if $t \geq 1 + \frac{2\ln(4)}{0.7 \cdot (0.89)^2 \cdot \mathbb{E}[X]}$. Since $\frac{2\ln(4)}{0.7 \cdot (0.89)^2} < 5$ and $\mathbb{E}[X] < 1$, so $t \geq \frac{6}{\mathbb{E}[X]}$ suffices. $\Pr[\mathcal{E}^{\text{normalval}}(t)] \geq 3/4$ follows similarly.

Next, observe that $\mathcal{E}^{\text{manyhigh}}(t) \cap \mathcal{E}^{\text{normalval}}(t)$ is independent of $\mathcal{E}^{\text{mistake}}$, since the two events depend on disjoint sets of independent random variables. Therefore, $\Pr[\mathcal{E}^{\text{mistake}} \cap \mathcal{E}^{\text{manyhigh}}(t) \cap \mathcal{E}^{\text{normalval}}(t)] = \Pr[\mathcal{E}^{\text{mistake}}] \cdot \Pr[\mathcal{E}^{\text{manyhigh}}(t) \cap \mathcal{E}^{\text{normalval}}(t)] \geq c \cdot 1/2$ for $t \geq 6/\,\mathbb{E}[X]$.

Let $\mathcal{E}_{\text{SPO}}(t)$ and $\mathcal{E}_{\text{EF}}(t)$ be the events that the allocation at step t is one-swap PO, and envy-free, respectively. When $\mathcal{E}^{\text{mistake}} \cap \mathcal{E}^{\text{manyhigh}}(t) \cap \mathcal{E}^{\text{normalval}}(t)$ occur, the allocation cannot be both one-swap

PO and envy-free, i.e. $\Pr\left[\overline{\mathcal{E}_{\mathrm{SPO}}(t) \cap \mathcal{E}_{\mathrm{EF}}(t)} \mid \mathcal{E}^{\mathrm{mistake}} \cap \mathcal{E}^{\mathrm{manyhigh}}(t) \cap \mathcal{E}^{\mathrm{normalval}}(t)\right] = 1$. To see this, notice that first, due to $\mathcal{E}^{\mathrm{mistake}}$, the only way to remain one-swap PO is to give each item j to agent 1 every time $\mathcal{E}_j$ occurs. Second, $\mathcal{E}^{\mathrm{manyhigh}}(t)$ ensures that agent 2's value for these items, and hence agent 2's value for agent 1's bundle, is at least $0.7 \cdot (t-1) \cdot \mathbb{E}[X] + v_{2,1}$. Third, $\mathcal{E}^{\mathrm{normalval}}(t)$ ensures that agent 2's value for all items is at most $1.1 \cdot (t-1) \cdot \mathbb{E}[X] + v_{2,1}$, which is strictly less than twice her value for agent 1's bundle. We conclude that the allocation at step t cannot be proportional, and is hence not envy-free. Overall, we have that

$$\begin{aligned}
\Pr\left[\overline{\mathcal{E}_{\mathrm{SPO}}(t)}\right] + \Pr\left[\overline{\mathcal{E}_{\mathrm{EF}}(t)}\right] &\geq \Pr\left[\overline{\mathcal{E}_{\mathrm{SPO}}(t)} \cup \overline{\mathcal{E}_{\mathrm{EF}}(t)}\right] \\
&= \Pr\left[\overline{\mathcal{E}_{\mathrm{SPO}}(t) \cap \mathcal{E}_{\mathrm{EF}}(t)}\right] \\
&\geq \Pr\left[\overline{\mathcal{E}_{\mathrm{SPO}}(t) \cap \mathcal{E}_{\mathrm{EF}}(t)} \cap \mathcal{E}^{\mathrm{mistake}} \cap \mathcal{E}^{\mathrm{manyhigh}}(t) \cap \mathcal{E}^{\mathrm{normalval}}(t)\right] \\
&= \Pr\left[\overline{\mathcal{E}_{\mathrm{SPO}}(t) \cap \mathcal{E}_{\mathrm{EF}}(t)} \mid \mathcal{E}^{\mathrm{mistake}} \cap \mathcal{E}^{\mathrm{manyhigh}}(t) \cap \mathcal{E}^{\mathrm{normalval}}(t)\right] \cdot \\
&\qquad \cdot \Pr[\mathcal{E}^{\mathrm{mistake}} \cap \mathcal{E}^{\mathrm{manyhigh}}(t) \cap \mathcal{E}^{\mathrm{normalval}}(t)] \\
&\geq c/2.
\end{aligned}$$

Therefore, for $t \geq 6/\mathbb{E}[X]$, at least one of $\Pr\left[\overline{\mathcal{E}_{\mathrm{SPO}}(t)}\right]$ and $\Pr\left[\overline{\mathcal{E}_{\mathrm{EF}}(t)}\right]$ is at least $c/4$. We conclude that no algorithm can be both envy-free and one-swap PO with high probability. □

Theorem 4 implies that when we have access to only ordinal information, we need to settle for *some* approximation to envy-freeness and efficiency. Our main positive result for this section is an algorithm that essentially matches the aforementioned lower bound.

Theorem 5. *In the i.i.d. model, Algorithm 1 achieves envy-freeness and a $(1-\varepsilon)$ approximation to welfare, with high probability, for all $\varepsilon > 0$.*

Algorithm 1 works in epochs: each epoch k has an exploration/sampling phase, where each agent i receives a pre-determined set of items, denoted G_i^k, irrespective of their valuation. This is followed by an exploitation/ranking phase, where each item is given to the agent with the highest empirical quantile (with respect to items received in the preceding exploration phase, i.e. G_i^k).

Algorithm 1: EF + $(1-\varepsilon)$-Welfare

```
for epoch k = 1 ... do
    Sampling Phase:   (n · k^4 items)
    Give the j-th item in this phase to agent j (mod n).
    Ranking Phase: (k^8 items)
    for each item g in this phase do
        Elicit σ_i^{-1}(G_i^k ∪ {g}, g) for all i ∈ N.
        Allocate g to an agent j ∈ arg min_{i∈N} σ_i^{-1}(G_i^k ∪ {g}, g).
```

We start with a technical lemma, which gives us a bound on the length of the exploration period we need in each epoch. The following definition will be useful.

Definition 2. A sample of $n \cdot m$ items (where each agent is allocated exactly m items) is ε-accurate if, with probability at least $1-\varepsilon$, the relative rank of a fresh item (with respect to the sample) is highest for the agent with highest quantile value.

Lemma 6. *If $\varepsilon, \delta \in (0,1)$, and $m \in \mathbb{Z}^+$ are such that $\varepsilon > 2n\sqrt{\frac{\ln(2n/\delta)}{2m}}$, then giving m samples to each agent is ε-accurate with probability at least $1-\delta$.*

Proof. We will use the Dvoretzky–Kiefer–Wolfowitz (DKW) inequality [DKW56, Mas90] to show the empirical CDF of sampled quantiles is reasonably close to a uniform distribution with probability $1-\delta$. We then show this is sufficient to guarantee ε-accuracy for the chosen ε. Let $\hat{F}_i$ be the empirical CDF of the sampled *quantiles* for agent i, i.e., $\hat{F}_i(q)$ for $q \in [0,1]$ is a random variable that describes the proportion of sampled items with quantile at most q. Note that $\hat{F}_i$ exactly captures agent i's ranking for a new item: if a fresh item has quantile q_i for agent i and q_j for agent j, then i ranks it higher than j exactly when $\hat{F}_i(q_i) > \hat{F}_j(q_j)$.

Noting that the CDF for the actual quantile distribution (i.e., the uniform distribution) is the identity on $[0,1]$, the DKW inequality states that for all $\gamma > 0$, $\Pr\left[\sup_{q\in[0,1]} |\hat{F}_i(q) - q| > \gamma\right] \leq 2e^{-2m\gamma^2}$. We want this condition to hold for all n agents, simultaneously, with probability at least $1-\delta$, so we pick γ such that $2e^{-2m\gamma^2} \leq \delta/n$ and apply a union bound; it suffices to choose $\gamma = \sqrt{\frac{\ln(2n/\delta)}{2m}}$.

We now show that the DKW condition ($\sup_{q\in[0,1]} |\hat{F}_i(q) - q| \leq \gamma$) being satisfied for all agents i is sufficient to guarantee ε-accuracy. Consider sampling quantiles $Q_1, \ldots, Q_n$ for a fresh item. Let $i^{\max} \in \operatorname{argmax}_{i\in\mathcal{N}} Q_i$ be a quantile-maximizing agent (technically a random variable). Our goal is to show that with probability at least $1-\varepsilon$ (with respect to the samples of $Q_1, \ldots, Q_n$) $\hat{F}_{i^{\max}}(Q_{i^{max}}) > \hat{F}_j(Q_j)$ for all $j \neq i^{\max}$. This ensures that $i^{\max}$ has the highest empirical rank, and hence receives the item. Let $Q_{(1)}, \ldots, Q_{(n)}$ be the respective order statistics. A key observation is that $Q_{(n)} - Q_{(n-1)} \sim \mathrm{Beta}[1,n]$ [Gen19]. The PDF of a $\mathrm{Beta}[1,n]$ distribution is $f(x) = nx^{n-1}$ for $x \in [0,1]$. Since $f(x) \leq n$, $\Pr\left[Q_{(n)} - Q_{(n-1)} < \rho\right] < n\rho$ for all $\rho > 0$. Plugging in $\rho = 2\gamma$, we have $\Pr\left[Q_{(n)} - Q_{(n-1)} \leq 2\gamma\right] < 2n\gamma$. We will show that as long as $\varepsilon > 2n\gamma$, ε-accuracy holds. First, we have $\Pr\left[Q_{(n)} - Q_{(n-1)} > 2\gamma\right] > 1-\varepsilon$. Conditioned on $Q_{(n)} - Q_{(n-1)} > 2\gamma$, the item is given to $i^{\max}$. To see why, observe $Q_{i^{\max}} = Q_{(n)}$ and $Q_j \leq Q_{(n-1)}$ for all $j \neq i^{\max}$, by definition. Using the DKW inequality condition, it follows that $\hat{F}_{i^{\max}}(Q_{i^{\max}}) \geq Q_{i^{\max}} - \gamma > Q_j + \gamma \geq \hat{F}_j(Q_j)$. We conclude that for $\varepsilon > 2n\sqrt{\frac{\ln(2n/\delta)}{2m}}$, ε-accuracy is satisfied with probability at least $1-\delta$. □

Using Lemma 6, we can get, for each epoch, a bound on the number of decisions where Algorithm 1 differs from the quantile maximization algorithm.

Lemma 7. *The allocation of Algorithm 1 differs from that of the quantile maximization algorithm after T steps by at most $f(T)$ items with high probability, where $f(T) \in O(poly(n) \cdot T^{15/16})$.*

Proof. We start by bounding the accuracy of Algorithm 1 in each epoch k. In epoch k, each agent receives k^4 items during the sampling phase. We claim that the sample in epoch k for $k \geq 3n$ is ε_k-accurate for $\varepsilon_k := 3n/k^{3/2}$ with probability at least $1-\delta_k$, for $\delta_k := 2n/e^{2k}$. Indeed, first note that by the choice of k, we have that $\varepsilon_k, \delta_k \in (0,1)$. Hence, we just need to show that these values satisfy the inequality of Lemma 6. We have that

$$\varepsilon_k = \frac{3n}{k^{3/2}} > \frac{2n}{k^{3/2}} = 2n\sqrt{\frac{1}{k^3}} = 2n\sqrt{\frac{\ln(e^{2k})}{2k^4}} = 2n\sqrt{\frac{\ln(2n/\delta_k)}{2k^4}}.$$

Next, fix a time T. Slightly abusing notation, let $k(t) = \min\{K \in \mathbb{N} | \sum_{k=1}^{K} nk^4 + k^8 \geq t\}$ be the function that given an item t returns the epoch item t is in. Notice that $T \geq \sum_{k=1}^{k(T)-1} nk^4 + k^8 \geq (k(T)-1)^8$, and therefore $k(T) \leq 2T^{1/8}$. In any run of the algorithm, we can classify every item $t \leq T$ into one of four categories.

1. Item t was allocated in one of the first $3n-1$ epochs, that is, $k(t) < 3n$.
2. Item t was allocated in the sampling phase of epoch $k(t) \geq 3n$.
3. Item t was allocated in the ranking phase of epoch $k(t) \geq 3n$; the epoch was $\varepsilon_{k(t)}$-accurate.
4. Item t was allocated in the ranking phase of epoch $k(t) \geq 3n$; the epoch was not $\varepsilon_{k(t)}$-accurate.

We say an item t was a mistake if it was given to an agent with a non-maximum quantile for it. We show that the number of mistakes in each category are bounded by $3^{10}n^9, 2nT^{5/8}, 9nT^{15/16}$, and $158n\ln(T)$ respectively, with high probability. This implies that the total number of mistakes is at most the sum of these quantities, which is $O(\text{poly}(n) \cdot T^{15/16})$, with high probability, via a union bound.

The number of items in the first category is at most

$$\sum_{k=1}^{3n-1} k^4 n + k^8 \leq \sum_{k=1}^{3n} (3n)^4 n + (3n)^8 \leq (3n)^5 n + (3n)^9 \leq 3^{10} n^9.$$

Hence, the number of mistakes in the first category is also at most $3^{10}n^9$.

For the second category, since $k(T) \leq 2T^{1/8}$, we have that the total number of items in the sampling phase is (with probability 1) upper bounded by

$$\sum_{k=1}^{k(T)} nk^4 \leq nk(T)^5 \leq 2nT^{5/8}. \qquad \square$$

Each item t in the third category has probability $\varepsilon_{k(t)}$ of being a mistake. The expected number of mistakes is therefore at most $\sum_{k=3n}^{k(T)} \varepsilon_{k(t)} k^8 = \sum_{k=3n}^{k(T)} 3nk^{13/2} \leq 3nk(T)^{15/2} \leq 8nT^{15/16}$. Using Hoeffding's inequality we get that with high probability the number of mistakes is at most $9nT^{15/16}$, since a deviation of $nT^{15/16}$ occurs with probability at most $\exp\left(-2n^2T^{15/8}/T\right) = \exp\left(-2n^2T^{7/8}\right)$.

For the fourth category, the expected number of items is at most $\sum_{k=3n}^{k(T)} \delta_k k^8 = 2n \sum_{k=3n}^{k(T)} \frac{k^8}{e^{2k}} \leq 2n \sum_{k=1}^{\infty} \frac{k^8}{e^{2k}} \leq 158n$. Using Markov's inequality we have that the number of mistakes is at most $158n\ln(T)$ with probability at least $1 - \ln(T)$, i.e., with high probability.

Finally, we can prove Theorem 5 as a relatively straightforward consequence of Lemma 7, since the ideal quantile maximization algorithm satisfies nice properties (e.g., Lemma 1).

Proof of Theorem 5. Fix a distribution D with CDF F and let X be a random variable with distribution D. Fix some ε to be $(1-\varepsilon)$-welfare-maximizing. Let $\mathcal{E}_1^T$ be the event that the maximum social welfare at time T is at least $1/2 \cdot \mathbb{E}[X] \cdot T$, let $\mathcal{E}_2^T$ be the event that quantile maximization is c-strongly-EF for $c = \frac{(\mathbb{E}[X \mid F(X) \geq 1/2] - \mathbb{E}[X])}{4n}$, and let $\mathcal{E}_3^T$ be the event that Algorithm 1 differs from quantile maximization on at most $f(T)$ items from Lemma 7. We first claim that $\mathcal{E}_1^T \cap \mathcal{E}_2^T \cap \mathcal{E}_3^T$ occurs with high probability in T. Note that Lemmas 1 and 7 tell us $\mathcal{E}_2^T$ and $\mathcal{E}_3^T$ each occur with high probability, respectively. For $\mathcal{E}_1^T$, the maximum value for each item is in expectation at least the expected value for a single agent $\mathbb{E}[X]$. Hence, a Chernoff bound tells us $\mathcal{E}_1^T$ occurs with probability at least $1 - \exp\left(\frac{-\mathbb{E}[X]T}{8}\right)$, i.e., with high probability. The claim holds because the intersection of a finite number of high probability events occurs with high probability.

Next, note that for sufficiently large T, since $f(T) \in o(T)$, $f(T) \leq \frac{(\mathbb{E}[X \mid F(X) \geq 1/2] - \mathbb{E}[X])}{8n} \cdot T$ and $f(T) \leq \varepsilon/2 \cdot \mathbb{E}[X] \cdot T$ (for any fixed ε that does not depend on T). Fix such a sufficiently large T. We show that, conditioned on $\mathcal{E}_1^T \cap \mathcal{E}_2^T \cap \mathcal{E}_3^T$, both EF and $(1-\varepsilon)$-welfare hold. Let $A^{QM} = (A_1^{QM}, \ldots, A_n^{QM})$ be the allocation of quantile maximization and $A = (A_1, \ldots, A_n)$ be the allocation of Algorithm 1. Beginning with envy-freeness, we have that for all pairs of agents i and j,

$$\begin{aligned}
v_i(A_i) &\geq^{(\mathcal{E}_3^T)} v_i(A_i^{QM}) - f(T) \\
&\geq^{(\mathcal{E}_2^T)} v_i(A_j^{QM}) - f(T) + \frac{(\mathbb{E}[X \mid F(X) \geq 1/2] - \mathbb{E}[X])T}{4n} \\
&\geq^{(\mathcal{E}_3^T)} v_i(A_j) - 2f(T) + \frac{(\mathbb{E}[X \mid F(X) \geq 1/2] - \mathbb{E}[X])T}{4n} \\
&\geq v_i(A_j),
\end{aligned}$$

so the allocation is envy-free. Further, noting that $\mathsf{sw}(A^{QM})$ is the maximum social welfare, we have the welfare approximation is at least

$$\begin{aligned}
\frac{\mathsf{sw}(A)}{\mathsf{sw}(A^{QM})} &= \frac{\mathsf{sw}(A^{QM}) - (\mathsf{sw}(A^{QM}) - \mathsf{sw}(A))}{\mathsf{sw}(A^{QM})} \\
&\geq^{(\mathcal{E}_3^T)} \frac{\mathsf{sw}(A^{QM}) - f(T)}{\mathsf{sw}(A^{QM})} \\
&= 1 - \frac{f(T)}{\mathsf{sw}(A^{QM})} \\
&\geq^{(\mathcal{E}_1^T)} 1 - \frac{f(T)}{1/2 \cdot \mathbb{E}[X] \cdot T} \\
&\geq^{(\mathcal{E}_3^T)} 1 - \frac{\varepsilon/2 \cdot \mathbb{E}[X] \cdot T}{1/2 \cdot \mathbb{E}[X] \cdot T} \\
&= 1 - \varepsilon,
\end{aligned}$$

as needed. □

5 Bounded memory in the i.i.d. model

In this section, we are interested in the more ambitious problem of designing dynamic algorithms with even more limited partial information: each agent is allowed to "remember" only a single item. We first show that, in this case, we need to settle for constant approximations of welfare.

Theorem 8. *In the i.i.d. model, given a memory of one item per agent, there is no algorithm $\mathcal{A}$ that is .999-welfare maximizing with high probability for all continuous and bounded value distributions.*

Proof. We prove that this negative result holds even for an even stronger class of algorithms in which, at each step t, the algorithm *selects* quantile thresholds $q_1^t, \ldots, q_n^t \in [0, 1]$ for each agent, and once an item arrives the algorithm observes, for each agent, whether the quantile of their sampled value $Q_{i,t}$ is above or below the threshold q_i^t. Note that this provides at least as much information about the fresh item as comparing it to any single prior item, since there is some uncertainty about the values and quantiles of all prior items.

We first focus on the algorithm for a single time-step and show there is a distribution of values such that, regardless of the quantile thresholds selected and allocations made, it cannot do well.

Fix a number of agents n and assume $n \geq 3$. We handle the special case of $n = 2$ at the end of this proof, as it requires a different distribution. For simplicity we consider a distribution that takes values larger than 1; re-scaling (specifically, dividing all values by $2 + \varepsilon$) gives a distribution upped bounded by 1 and does not affect any of our arguments. Consider the value distribution X, with

$$X \sim \begin{cases} \text{Unif}[0, \varepsilon] & \text{with probability } 1 - \frac{1}{n}, \\ \text{Unif}[1, 1 + \varepsilon] & \text{with probability } \frac{2}{3n}, \text{and} \\ \text{Unif}[2, 2 + \varepsilon] & \text{with probability } \frac{1}{3n} \end{cases}$$

for some small $\varepsilon > 0$ to be fixed later. Intuitively, X is a continuous version of a discrete distribution which takes low value (near 0) with probability $1 - \frac{1}{n}$, medium value (near 1) with probability $\frac{2}{3n}$, and high value (near 2) with probability $\frac{1}{3n}$. Let F_X be its CDF. Trivially, the maximum social welfare of T items when all agents have this value distribution is at most $T \cdot (2 + \varepsilon)$.

We show that regardless of what quantile thresholds the algorithm chooses at step t and which decision it makes given the resulting signals, the expected value of the agent receiving item t is at least $(1 - \varepsilon) \cdot \frac{1}{144e}$ away from optimal. To that end, fix arbitrary thresholds $q_1, \ldots, q_n$. First, we partition the agents depending on whether their quantile q_i is above or below $1 - \frac{2n}{3}$. We let $N^{\text{below}} = \{\, i \in [n] \mid q_i < 1 - \frac{2n}{3} \,\}$ and $N^{\text{above}} = \{\, i \in [n] \mid q_i \geq 1 - \frac{2n}{3} \,\}$. Either $|N^{\text{below}}| \geq \lceil n/2 \rceil$ or $|N^{\text{above}}| \geq \lceil n/2 \rceil$; we analyse each case separately. Since $n \geq 3$, we have $\lceil n/2 \rceil \geq 2$.

Case I: $|N^{below}| \geq \lceil n/2 \rceil$. In this case, it will be difficult for the algorithm to distinguish between agents in N^{below} with medium value and those with high value. Consider the event $\mathcal{E}$ that one agent $i^{\max} \in N^{\text{below}}$ has quantile $Q_{i^{\max}} > 1 - \frac{1}{3n}$, one agent $i^{\text{smax}} \in N^{\text{below}}$ has quantile $Q_{i^{\text{smax}}} \in (1 - \frac{2}{3n}, 1 - \frac{1}{3n})$, and all other agents $i \in \mathcal{N} \setminus \{\, i^{\max}, i^{\text{smax}} \,\}$ have quantile $Q_i < 1 - \frac{1}{n}$. First, we show that $\Pr[\mathcal{E}] \geq \frac{1}{72e}$, a constant. To compute this probability, note that there are at least $\lceil n/2 \rceil \cdot (\lceil n/2 \rceil - 1)$ choices of $i^{\max}$ and i^{smax}. Once these have been selected, the probability of $\mathcal{E}$ occurring for this pair of agents is

$$\frac{1}{3n} \cdot \frac{1}{3n} \cdot \left(1 - \frac{1}{n}\right)^{n-2} \geq^{(n \geq 3)} \frac{1}{9n^2} \left(1 - \frac{1}{n}\right)^{n-1} \geq \frac{1}{9en^2}.$$

Since $\lceil n/2 \rceil \cdot (\lceil n/2 \rceil - 1) \geq n^2/8$, we can that conclude $\Pr[\mathcal{E}] \geq \frac{1}{72e}$. Conditioned on $\mathcal{E}$ occurring, $i^{\max}$ has high value, i^{smax} has medium value, and all other agents have low value. However, from the perspective of the algorithm, two agents ($i^{\max}$ and i^{smax}) give a high signal, and it's equally likely that each of them is the agent with the high value (note that we condition on $\mathcal{E}$). The algorithm must therefore allocate the item to an agent with at most medium value (upper bounded by $1 + \varepsilon$) with probability at least $1/2$, even though an agent with value at least 2 exists. Hence, in this timestep, the algorithm has an additive error (compared to the optimum welfare) of at least $(1 - \varepsilon)$ with probability at least $\frac{1}{144e}$.

Case II: $|N^{above}| \geq \lceil n/2 \rceil$. In this case, it will be difficult for the algorithm to distinguish between agents in N^{above} that have medium value and those with low value. Consider the event $\mathcal{E}$

Algorithm 2: Bounded Memory

```
for Epoch k = 1 ... do
    Sampling Phase: (k^9 items)
    NotWithinError ← 𝒩
    for trial = 1, ..., k^3 do
        for i ∈ NotWithinError do
            Allocate the next item to agent i, and update her memory
        Test k^6 − |NotWithinError| number of items (for each agent)
        for i ∈ NotWithinError do
            if Proportion of test items for agent i is within ±1/k^2 of (n − 1)/n then
                NotWithinError ← NotWithinError \ { i }
    Ranking Phase: (k^18 items)
    for each item g in this phase do
        if Some agent i has high signal then
            Give g to a (uniformly) random such agent
        else
            Give g to an agent uniformly at random
```

that one agent $i^{\max} \in N^{\text{above}}$ has quantile $Q_{i^{\max}} \in (1-\frac{1}{n}, 1-\frac{2}{3n})$ and all other agents $i \in \mathcal{N} \setminus \{ i^{\max} \}$ have quantile $Q_i < 1 - \frac{1}{n}$. First, we show that $\Pr[\mathcal{E}] \geq \frac{1}{6e}$. Indeed, there are at least $n/2$ choices for $i^{\max}$. For a fixed choice of $i^{\max}$, the probability of $\mathcal{E}$ occurring is $\frac{1}{3n} \cdot \left(1 - \frac{1}{n}\right)^{n-1} \geq \frac{1}{3en}$, and there are at least $n/2$ choices for $i^{\max}$, so $\Pr[\mathcal{E}] \geq \frac{1}{6e}$. Agent $i^{\max}$ and the other members of N^{above} (there is at least one more) are indistinguishable to the algorithm as they all have a low signal, so the algorithm must give it to an agent with value at most ε with probability at least 1/2 even though an agent with value at least 1 exists. Hence, in this timestep, the algorithm has an additive error (compared to the optimum welfare) of at least $(1 - \varepsilon)$ with probability at least $\frac{1}{12e}$.

In either case, for every time step, the algorithm has an additive error of at least $(1 - \varepsilon)$ with probability at least $\frac{1}{144e}$, irrespective of the past allocations. As time steps are independent, standard tail bounds give that, for sufficiently small $\varepsilon > 0$, the error is at least $\frac{1-\varepsilon}{1000}T$ with high probability. The optimal social welfare is at most $(2 + \varepsilon) \cdot T$; we conclude the algorithm can be no more than an 0.999−approximation to welfare.

The case of two agents. Finally, we handle the case of two agents. Assume values are drawn from a Unif$[0, 1]$ distribution. Let q_1, q_2 be the quantile thresholds selected by the algorithm and, without loss of generality, suppose that $0 \leq q_1 \leq q_2 \leq 1$. At least one of the differences $q_1 - 0, q_2 - q_1, 1 - q_2$ must be at least 1/3. Suppose $q_2 - q_1 \geq 1/3$ (the other cases are symmetric). We investigate the event that both agents have $Q_i \in [q_1, q_2]$, so that agent 1 signals high and agent 2 signals low, which occurs with probability at least 1/9. Conditioned on this event, the signals do not provide any additional information, so the algorithm chooses the agent with smaller value at least half of the time. In this case, the expected difference between the larger and smaller values is 1/9. Hence, the expected difference of the value from the algorithm versus the maximum social welfare is at least $\frac{1}{9} \cdot \frac{1}{2} \cdot \frac{1}{9} = 1/162$ on each item. The maximum social welfare is at most T, and we expect the difference to be at least $T/1000$ due to concentration, so the algorithm cannot guarantee more than a .999 approximation, as needed. □

Our positive result matches this lower bound up to a constant.

Theorem 9. *In the i.i.d. model, given a memory of one item per agent, Algorithm 2 achieves envy-freeness and a* $(1-1/e)^2 - \varepsilon$ *approximation to welfare, with high probability, for all* $\varepsilon > 0$.

Algorithm 2 works in epochs, similar to Algorithm 1. In each epoch's exploration/sampling phase, it tries to find an item whose quantile is close to the $\frac{n-1}{n}$-threshold algorithm. Epoch k makes k^3 such attempts, and each candidate item is tested against k^6 fresh items to get an estimated quantile. If everything is within the error we can tolerate, the algorithm remembers this item for this epoch; otherwise, the agent has an arbitrary item in memory during this epoch. During the exploitation/ranking phase, Algorithm 2 tries to mimic the $\frac{n-1}{n}$-threshold algorithm (instead of the quantile maximization algorithm as Algorithm 1 did), and, in fact, inherits its approximation factor (Lemma 2) exactly.

Our first technical lemma, Lemma 10, gives necessary bounds on the various variables of Algorithm 2 for a sample to be ε-accurate with respect to the ideal threshold algorithm; see Definition 3. Its proof can be found in Appendix A.2.

Definition 3. A set of n items in memory, one for each agent, is ε-accurate with respect to q^* if with probability at least $1-\varepsilon$, when a fresh item is sampled, the agents with true quantile above q^* are exactly those that value the fresh item more than their item in memory.

Lemma 10. *For all* $\varepsilon, \delta \in (0,1)$*, if (1) at least* τ *trials are done with* $\tau \geq \frac{\ln(2n/\delta)}{\varepsilon/(3n)}$*, and (2) at least* ℓ *test items are used per trial for* $\ell \geq \frac{18n^2}{\varepsilon^2} \ln\left(\frac{4\tau n}{\delta}\right)$*, and (3) the tolerance for accepting an item is* $\varepsilon/(3n)$*, then the items in memory are* ε*-accurate (for all agents, simultaneously) with respect to* $q^* = \frac{n-1}{n}$*, with probability at least* $1-\delta$.

Though Lemmas 6 and 10 resemble each other (and are used in analogous ways), the proofs require different techniques, as the sampling processes are very different. Next, we prove an analogue to Lemma 7: the number of disagreements between Algorithm 2 and the ideal threshold algorithm is sublinear.[2] The proofs of Lemmas 7 and 11 are similar, precisely because Lemma 6 matches Lemma 10. Theorem 9 follows from Lemma 11 as in the i.i.d. case. The proofs of Lemma 11 and Theorem 9 can be found in Appendices A.3 and A.4 respectively.

Lemma 11. *The allocation of Algorithm 2 differs from that of the* $\frac{n-1}{n}$*-threshold algorithm after* T *steps by at most* $f(T)$ *items with high probability, where* $f(T) \in O(poly(n) \cdot T^{17/18})$.

6 The non-i.i.d. model

In this section, we study the non-i.i.d. model. We first establish a strong lower bound for the non-i.i.d. model. The following negative result holds even for algorithms that know the associated quantile for every fresh item.

Theorem 12. *Even for* 2 *non-identical agents, there is no algorithm that is EF and* c*-PO with probability* p*, for* $c > \frac{1+\sqrt{5}}{4} \approx .809$ *and* $p > 2/3$*, for all continuous and bounded value distributions.*

Proof. Suppose for contradiction that there is an algorithm $\mathcal{A}$ so that for all bounded continuous distributions (X_1, X_2) there exists a $T^* = T^*(X_1, X_2)$ where for all $t \geq T^*$, $\mathcal{A}$ is envy-free and c-PO

[2] Note these are randomized algorithms, so by "differ on a item" here we mean that the distributions over agents receiving the item differ.

with probability p with $p > 2/3$ for some constant $c > \frac{1+\sqrt{5}}{4}$. Hence, there is some ε such that $p > 2/3 + \varepsilon$ and $1/c < \frac{4}{1+\sqrt{5}} - \varepsilon = \sqrt{5} - 1 - \varepsilon$.

Consider two distributions D_F and D_S; we describe these later in the proof. Consider the three instances $I_0 = (D_F, D_F), I_1 = (D_S, D_F)$ and $I_2 = (D_F, D_S)$.

Let $\mathcal{E}_j^{\mathcal{A},t}$ be the event that $\mathcal{A}$ is envy-free and c-PO on instance I_j at time t for $j \in \{0, 1, 2\}$. By construction, $\Pr\left[\mathcal{E}_j^{\mathcal{A},t}\right] \geq 2/3 + \varepsilon$ for all $j \in \{0, 1, 2\}$ and $t \geq T^*$.

Let z be a parameter we will fix later in the proof, and let $Z_i^t = \mathbb{I}\{Q_{i,t} \geq 1 - z\}$ for $i = \{1, 2\}$. Observe that $Z_1^t \cdot Z_2^t$ is 1 with probability z^2 and 0 otherwise. The following events characterize a specific notion of a "nice" sample, in which the number of items with high quantiles for both agents is near its expectation: $\mathcal{E}_1^T = \mathbb{I}\{|\frac{1}{T}\sum_{t=1}^T Z_1^t \cdot Z_2^t - z^2| < \delta\}$, $\mathcal{E}_2^T = \mathbb{I}\{|\frac{1}{T}\sum_{t=1}^T Z_1^t - z| < \delta\}$, and $\mathcal{E}_3^T = \mathbb{I}\{|\frac{1}{T}\sum_{t=1}^T Z_2^t - z| < \delta\}$ for some $\delta > 0$. By Hoeffding's inequality, $\Pr\left[\bar{\mathcal{E}}_1^T\right] = \Pr\left[|\frac{1}{T}\sum_{t=1}^T Z_1^t \cdot Z_2^t - z^2| \geq \delta\right] \leq 2\exp\left(-2T\delta^2\right)$. It follows that for $T \geq \log(2/\varepsilon)/(2\delta^2)$, $\Pr\left[\bar{\mathcal{E}}_1^T\right] \leq \varepsilon$. Similarly, for $T \geq \log(2/\varepsilon)/(2\delta^2)$, it holds that $\Pr\left[\bar{\mathcal{E}}_2^T\right] \leq \varepsilon$, and $\Pr\left[\bar{\mathcal{E}}_3^T\right] \leq \varepsilon$. Consider an arbitrary $T > T_{\max} = \max\{T_0, T_1, T_2, \log(2/\varepsilon)/(2\delta^2)\}$. Applying a union bound,

$$\Pr\left[\bar{\mathcal{E}}_0^{\mathcal{A},T} \cup \bar{\mathcal{E}}_1^{\mathcal{A},T} \cup \bar{\mathcal{E}}_2^{\mathcal{A},T} \cup \bar{\mathcal{E}}_1^T \cup \bar{\mathcal{E}}_2^T \cup \bar{\mathcal{E}}_3^T\right] \leq \sum_{i=0}^{2} \Pr\left[\bar{\mathcal{E}}_i^{\mathcal{A},T}\right] + \sum_{i=1}^{3} \Pr\left[\bar{\mathcal{E}}_i^T\right] < 3 \cdot (\frac{1}{3} - \varepsilon) + 3\varepsilon = 1.$$

It follows that $\Pr\left[\mathcal{E}_0^{\mathcal{A},T} \cap \mathcal{E}_1^{\mathcal{A},T} \cap \mathcal{E}_2^{\mathcal{A},T} \cap \mathcal{E}_1^T \cap \mathcal{E}_2^T \cap \mathcal{E}_3^T\right] > 0$. Therefore, there must exist a sequence of T items whose quantiles satisfy all of $\mathcal{E}_1^T$, $\mathcal{E}_2^T$, and $\mathcal{E}_3^T$, and, since $\mathcal{A}$ does not have access to the items' values, there must exist an allocation A^T for these T items (in the support of $\mathcal{A}$) that is EF and c-PO, no matter which of I_0, I_1 or I_2 the values were taken from. Let $q^T = \{(q_1(t), q_2(t))\}_{t=1}^T$ be these items' quantiles. Let $H_B = \{t \in [T] : q_1(t) \geq 1 - z \text{ and } q_2(t) \geq 1 - z\}$ be the items for which $Z_1^t \cdot Z_2^t = 1$, and $H_1 = \{t \in [T] : q_1(t) \geq 1 - z\}$ the items for which $Z_1^t = 1$.

Set distributions $D_F = \text{Unif}[1-w, 1]$ and D_S, under which each item is $\text{Unif}[0, w]$ with probability z and at $\text{Unif}[1 - w, 1]$ with probability $1 - z$, for some small positive w that we fix later in the proof.

We have that some agent receives at most half the items in H_B; without loss of generality this is agent 2, i.e., $|A_2^T \cap H_B| \geq |H_B|/2$. We show that there exists a feasible more than $1/c$ Pareto-improvement under the values in I_1. To that end, we compare A^T to the allocation $\hat{A}$ where $\hat{A}_1 = H_1$ and $\hat{A}_2 = \bar{H}_1$.

We next bound the utilities of each agent under A^T and $\hat{A}$. Beginning with agent 1, we have

$$\begin{aligned}
u_1(\hat{A}_1) &= u_1(H_1) \\
&\geq |H_1| \cdot (1 - w) \\
&\geq^{(\mathcal{E}_1^T)} T \cdot (z - \delta)(1 - w) \\
&= T(z - \delta - zw + \delta w) \\
&\geq T(z - \delta - w)
\end{aligned}$$

and

$$\begin{aligned}
u_1(A_1) &\leq w \cdot |A_1 \cap \bar{H}_1| + 1 \cdot |A_1 \cap H_1| \\
&\leq T \cdot w + |H_1| - |A_2 \cap H_1|
\end{aligned}$$

$$
\begin{aligned}
&\leq T \cdot w + |H_1| - |A_2 \cap H_B| \\
&\leq^{(\mathcal{E}_2^T)} T \cdot w + T(z+\delta) - |A_2 \cap H_B| \\
&\leq T \cdot w + T(z+\delta) - |H_B|/2 \\
&\leq^{(\mathcal{E}_1^T)} T \cdot w + T(z+\delta) - T(z^2-\delta)/2 \\
&= T(z - z^2/2 + w + 3\delta/2).
\end{aligned}
$$

Together, these imply

$$
\frac{u_1(\hat{A}_1)}{u_1(A_1^T)} \geq \frac{z-\delta-w}{z-z^2/2+w+3\delta/2} = \frac{2z-2\delta-2w}{2z-z^2+2w+3\delta}.
$$

Next, we consider agent 2. We have

$$
\begin{aligned}
u_2(\hat{A}_2) &= u_2(\bar{H}_1) \\
&\geq (1-w)|\bar{H}_1| \\
&= (1-w)(T-|H_1|) \\
&\geq^{(\mathcal{E}_2^T)} (1-w)T \cdot (1-(z+\delta)) \\
&= T(1-z-\delta-w+wz+w\delta) \\
&\geq T(1-z-\delta-w).
\end{aligned}
$$

By $\mathcal{E}_0^{\mathcal{A}}$, A^T is envy-free on I_0. It follows that $|A_1^T| \geq (1-w)|A_2^T|$. Since $|A_1^T| + |A_2^T| = T$, we have that $|A_2^T| \leq \frac{1}{2-w}T$. Hence, $u_2(A_2^T) \leq |A_2^T| \leq \frac{1}{2-w}T$. Combining these, we have

$$
\frac{u_2(\hat{A}_2)}{u_2(A_2^T)} = \frac{1-z-\delta-w}{\frac{1}{2-w}} = 2-2z-2\delta-2w-w+wz+w\delta+w^2 \geq 2-2z-2\delta-3w.
$$

Choose $z = \frac{3-\sqrt{5}}{2}$. Note that $z^2 = \frac{7-3\sqrt{5}}{2}$. Choose $\delta, w < \varepsilon/25$. We then have,

$$
\begin{aligned}
\frac{u_1(\hat{A}_1)}{u_1(A_1^T)} &> \frac{3-\sqrt{5}-\varepsilon/5}{(\sqrt{5}-1)/2+\varepsilon/5} \\
&= \frac{3-\sqrt{5}}{(\sqrt{5}-1)/2+\varepsilon/5} - \frac{\varepsilon/5}{(\sqrt{5}-1)/2+\varepsilon/5} \\
&> \frac{3-\sqrt{5}}{(\sqrt{5}-1)/2+\varepsilon/5} - \frac{2\varepsilon}{5} && \left(\frac{\sqrt{5}-1}{2} + \frac{\varepsilon}{5} > 1/2\right) \\
&> \frac{3-\sqrt{5}}{(\sqrt{5}-1)/2 \cdot (1+2\varepsilon/5)} - \frac{2\varepsilon}{5} && (\sqrt{5}-1 > 1) \\
&= (\sqrt{5}-1) \cdot \frac{1}{1+2\varepsilon/5} - \frac{2\varepsilon}{5} \\
&> (\sqrt{5}-1) \cdot (1-2\varepsilon/5) - \frac{2\varepsilon}{5} \\
&> (\sqrt{5}-1) - \varepsilon/2 - \frac{2\varepsilon}{5} && ((\sqrt{5}-1) \cdot 2/5 < 1/2) \\
&> \sqrt{5}-1-\varepsilon
\end{aligned}
$$

$$> 1/c$$

and

$$\frac{u_2(\hat{A}_2)}{u_2(A_2^T)} > 2 - (3 - \sqrt{5}) - \varepsilon/5 > \sqrt{5} - 1 - \varepsilon > 1/c,$$

so this is more than a $1/c$ Pareto Imrovement. □

Algorithms 1 and 2 are envy-free with high probability, even in the non-i.i.d. model, since envy-freeness is not an "inter-agent" property. Our last result shows that they also give a constant approximation to Pareto efficiency, by combining Lemma 3 with Lemmas 7 and 11. Its proof can be found in Appendix A.5.

Theorem 13. *In the non-i.i.d. model, both Algorithm 1 (unbounded memory) and Algorithm 2 (one-item memory) are EF and $(1/e - \varepsilon)$-PO, with high probability, for all $\varepsilon > 0$.*

Interestingly, the guarantees for Algorithm 2 in the non-i.i.d. model are only marginally worse compared to the i.i.d. model; the approximation ratio decreases from $(1 - 1/e)^2 \approx 0.4$ to $1/e \approx 0.37$. Finally, we note that, even though the formal guarantees in Theorem 13 are the same for the two algorithms, and even though Algorithm 2 uses memory size of one, Algorithm 1 has the benefit of much shorter epoch lengths (in addition to better guarantees under the i.i.d. model).

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Appendix

A Missing Proofs

A.1 Proof of Lemma 1

We focus on quantile maximization. The same proof goes through essentially unchanged for the threshold algorithm; we explain the differences whenever it's appropriate.

Fix distributions $D_1, \dots, D_n$ with CDFs $F_1, \dots, F_n$. Fix two agents i and j. We will show with high probability, i does not envy j (in a strong sense). Union bounding over all $\binom{n}{2}$ pairs yields the lemma statement.

As in [DGK+14], we compare the expected contribution of an item t to i's bundle and its expected contribution to j's bundle. Let A be the random variable denoting the agent that received the item. We want to consider the difference $\mathbb{E}[v_{i,t} \cdot \mathbb{I}[A = i]] - \mathbb{E}[v_{i,t} \cdot \mathbb{I}[A = j]]$. Let H_i be the event that $F_i(v_{i,t}) \geq \frac{n-1}{n}$, and L_i be the compliment. We split each of the two terms into conditional expectations depending on the signal, beginning with the first one:

$$\mathbb{E}[v_{i,t} \cdot \mathbb{I}[A = i]] = \mathbb{E}\left[v_{i,t} \cdot \mathbb{I}[A = i] \mid H_i\right] \cdot \Pr[H_i] + \mathbb{E}\left[v_{i,t} \cdot \mathbb{I}[A = i] \mid L_i\right] \cdot \Pr[L_i].$$

Note that, under quantile maximization, $v_{i,t}$ is positively correlated with $\mathbb{I}[A = i]$: for any fixed value $v_{i,t}$, $A = i$ with probability $F(v_{i,t})^{n-1}$, which is increasing in $v_{i,t}$. Therefore, the expectation of the product is greater than or equal to the product of the expectations: $\mathbb{E}\left[v_{i,t} \cdot \mathbb{I}[A = i] \mid H_i\right] \geq \mathbb{E}\left[v_{i,t} \mid H_i\right] \cdot \Pr[A = i \mid H_i]$ and $\mathbb{E}\left[v_{i,t} \cdot \mathbb{I}[A = i] \mid L_i\right] \geq \mathbb{E}\left[v_{i,t} \mid L_i\right] \cdot \Pr[A = i \mid L_i]$. Therefore

$$\begin{aligned}\mathbb{E}[v_{i,t} \cdot \mathbb{I}[A = i]] &\geq \mathbb{E}\left[v_{i,t} \mid H_i\right] \cdot \Pr[A = i \mid H_i] \cdot \Pr[H_i] + \mathbb{E}\left[v_{i,t} \mid L_i\right] \cdot \Pr[A = i \mid L_i] \cdot \Pr[L_i] \\ &= \mathbb{E}\left[v_{i,t} \mid H_i\right] \cdot \Pr[A = i \text{ and } H_i] + \mathbb{E}\left[v_{i,t} \mid L_i\right] \cdot \Pr[A = i \text{ and } L_i].\end{aligned}$$

For the threshold algorithm, we have equality above, since conditioned on either H_i or L_i, $v_{i,t}$ is independent of $\mathbb{I}[A = i]$, as the allocation depends only on the high vs low signal.

On the other hand, $v_{i,t}$ is negatively correlated with $\mathbb{I}[A = j]$. Therefore

$$\mathbb{E}[v_{i,t} \cdot \mathbb{I}[A = j]] \leq \mathbb{E}\left[v_{i,t} \mid H_i\right] \cdot \Pr[A = j \text{ and } H_i] + \mathbb{E}\left[v_{i,t} \mid L_i\right] \cdot \Pr[A = j \text{ and } L_i].$$

Again, for the threshold algorithm, we have equality.

Combined, we have

$$\begin{aligned}\mathbb{E}[v_{i,t} \cdot \mathbb{I}[A = i]] - \mathbb{E}[v_{i,t} \cdot \mathbb{I}[A = j]] &\geq \mathbb{E}\left[v_{i,t} \mid H_i\right] \cdot \left(\Pr[A = i \text{ and } H_i] - \Pr[A = j \text{ and } H_i]\right) && (1) \\ &\quad - \mathbb{E}\left[v_{i,t} \mid L_i\right] \cdot \left(\Pr[A = j \text{ and } L_i] - \Pr[A = i \text{ and } L_i]\right). && (2)\end{aligned}$$

We analyze (1), $\Pr[A = i \text{ and } H_i] - \Pr[A = j \text{ and } H_i]$. Let H_j be the event that $F_j(v_{j,t}) \geq \frac{n-1}{n}$ Let L_j be its complement. We have:

$$\begin{aligned}&\left(\Pr[A = i \text{ and } H_i \text{ and } L_j] + \Pr[A = i \text{ and } H_i \text{ and } H_j]\right) \\ &\quad - \left(\Pr[A = j \text{ and } H_i \text{ and } L_j] + \Pr[A = j \text{ and } H_i \text{ and } H_j]\right).\end{aligned}$$

Notice that $\Pr[A = j \text{ and } H_i \text{ and } L_j] = 0$ because if agent i has a high quantile and j has a low quantile, j cannot receive the item (in either algorithm). Additionally, by symmetry, $\Pr[A = i \text{ and } H_i \text{ and } H_j] =$

$\Pr[A = j \text{ and } H_i \text{ and } H_j]$. Therefore, (1) simplifies to $\Pr[A = i \text{ and } H_i \text{ and } L_j]$. Finally, we note that $\Pr[A = i \text{ and } H_i \text{ and } L_j] \geq \frac{1}{n-1}$, again, for both algorithms.

We analyze (2), $\Pr[A = j \text{ and } L_i] - \Pr[A = i \text{ and } L_i]$. Let $\mathcal{E}^{\text{low}}$ be the event that all agents other than i have quantile lower then $\frac{n-1}{n}$. Let $\overline{\mathcal{E}^{\text{low}}}$ be its complement, the probability that at least one agent other than i has a high quantile. We have:

$$\left(\Pr\left[A = j \text{ and } L_i \text{ and } \mathcal{E}^{\text{low}}\right] + \Pr\left[A = j \text{ and } L_i \text{ and } \overline{\mathcal{E}^{\text{low}}}\right]\right) - \left(\Pr\left[A = i \text{ and } L_i \text{ and } \mathcal{E}^{\text{low}}\right] + \Pr\left[A = i \text{ and } L_i \text{ and } \overline{\mathcal{E}^{\text{low}}}\right]\right).$$

Notice that $\Pr\left[A = i \text{ and } L_i \text{ and } \overline{\mathcal{E}^{\text{low}}}\right] = 0$ because if agent i has a low quantile and at least one other agent has a high quantile, i cannot receive the item. Additionally, by symmetry,

$$\Pr\left[A = j \text{ and } L_i \text{ and } \mathcal{E}^{\text{low}}\right] = \Pr\left[A = i \text{ and } L_i \text{ and } \mathcal{E}^{\text{low}}\right],$$

since when all agents have low quantiles, i and j are equally likely to receive the item. Hence, the probability in (2) simplifies to $\Pr\left[A = j \text{ and } L_i \text{ and } \overline{\mathcal{E}^{\text{low}}}\right] = \Pr\left[A = j | L_i \text{ and } \overline{\mathcal{E}^{\text{low}}}\right] \cdot \Pr\left[L_i \text{ and } \overline{\mathcal{E}^{\text{low}}}\right]$. The first term is equal to $1/(n-1)$, since j is equally likely to receive the item compared to any agent. The second term is $\frac{n-1}{n} \cdot \left(1 - \left(\frac{n-1}{n}\right)^{n-1}\right)$. Observing that $\left(\frac{n-1}{n}\right)^{n-1} = \left(1 - \frac{1}{n}\right)^{n-1} \geq \frac{1}{e}$, we have that the probability in (2) is at most $\frac{1}{n}(1 - \frac{1}{e})$.

Overall, we have shown that

$$\begin{aligned}
\mathbb{E}[v_{i,t} \cdot \mathbb{I}[A = i]] - \mathbb{E}[v_{i,t} \cdot \mathbb{I}[A = j]] &\geq \mathbb{E}\left[v_{i,t} \mid H_i\right] \frac{1}{n-1} - \mathbb{E}\left[v_{i,t} \mid L_i\right] \frac{e-1}{en} \\
&\geq \frac{e-1}{en} \left(\mathbb{E}\left[v_{i,t} \mid H_i\right] - \mathbb{E}\left[v_{i,t} \mid L_i\right]\right) \\
&\geq \frac{1}{2n} \left(\mathbb{E}\left[v_{i,t} \mid H_i\right] - \mathbb{E}\left[v_{i,t} \mid L_i\right]\right) \\
&= \frac{1}{2n} \left(\mathbb{E}\left[X_i \mid H_i\right] - \mathbb{E}\left[X_i \mid L_i\right]\right)
\end{aligned}$$

It remains to show that the value of i for A_i^T is at least her value for A_j^T plus $\frac{1}{4n}\left(\mathbb{E}\left[X_i \mid H_i\right] - \mathbb{E}\left[X_i \mid L_i\right]\right)$ with high probability. Towards this, notice that the value of i for A_i^T minus her value for A_j^T is the sum of T i.i.d. random variables, supported in $[-1, 1]$, whose expectation is at least $\frac{1}{2n}\left(\mathbb{E}\left[X_i \mid H_i\right] - \mathbb{E}\left[X_i \mid L_i\right]\right)$, as we've established so far. Hoeffding's inequality then implies that the probability that this difference is less than $b = \frac{1}{4n}\left(\mathbb{E}\left[X_i \mid H_i\right] - \mathbb{E}\left[X_i \mid L_i\right]\right)$ is at most $2\exp\left(-\frac{b^2 T}{2}\right)$, i.e., exponentially small, since b is a constant. Observing that $\mathbb{E}\left[X_i \mid H_i\right] \geq \mathbb{E}\left[X_i \mid Q_i \geq 1/2\right]$ and $\mathbb{E}\left[X_i \mid L_i\right] \leq \mathbb{E}[X_i]$ concludes the proof. □

A.2 Proof of Lemma 10

Fix such an ε, δ, τ, and ℓ. We claim that a sufficient condition for ε-accuracy is that all agents accept an item with quantile within $q^* \pm \varepsilon/(2n)$. Indeed, note that any sampled quantile outside this range will be classified (as high vs low) correctly. With such an error tolerance, the probability a specific agent's quantile (for a fresh item) falls within this range is at most ε/n. Via a union bound over all n agents, the probability that *no* agent has a quantile (for a fresh item) within this range is at least $1-\varepsilon$. Hence, all that needs to be shown is that with probability $1-\delta$, all agents accept an item and the accepted item has quantile within the allowed range.

Since there are τ trials, there are at most $n\tau$ items tested across all agents. We show that ℓ is large enough such that with probability $1-\delta/2$, all these tests are within $\pm\varepsilon/(6n)$ of the true value. Using Hoeffding's inequality, the probability any specific test fails is at most

$$2\exp\left(-2\left(\frac{\varepsilon}{6n}\right)^2\cdot\ell\right) = 2\exp\left(-\frac{\varepsilon^2}{18n^2}\cdot\ell\right) \le 2\exp\left(-\ln\left(\frac{4\tau n}{\delta}\right)\right) = \frac{\delta}{2\tau n},$$

a union bound over all $n\tau$ tests yields the required probability.

Note that under the condition that all the tests are this accurate, since the threshold for acceptance is $\pm\varepsilon/(3n)$, any accepted item will be within $\pm\varepsilon/(2n)$ of q^*, as needed. What remains to be shown is that each agent will, with reasonable probability, accept an item. To that end, we need to show that with probability $1-\delta/2$, all agents will test an item that is within $\pm\varepsilon/(6n)$ of q^*. If such an item is tested and the test is accurate, the empirical estimate of its quantile is within $\pm\varepsilon/(3n)$, and the item would hence be accepted. A union bound will then tell us that both of these events would occur with probability $1-\delta$.

Towards proving that each agent will test an item within $\pm\varepsilon/(6n)$ of q^* with probability $1-\delta/2$, we use a union bound, showing that each agent individually will *not* sample such an item with probability at most $\delta/(2n)$. In each of τ trials, the probability such an item is sampled is $\varepsilon/(3n)$. Hence, the probability no such item is sampled is $\left(1-\frac{\varepsilon}{3n}\right)^\tau$. We then have that

$$\begin{aligned}
\left(1-\frac{\varepsilon}{3n}\right)^\tau &= \left(1-\frac{1}{\frac{3n}{\varepsilon}}\right)^\tau \\
&= \left(\left(1-\frac{1}{\frac{3n}{\varepsilon}}\right)^{\frac{3n}{\varepsilon}}\right)^{\tau\cdot\frac{\varepsilon}{3n}} \\
&\le \left(e^{-1}\right)^{\tau\cdot\frac{\varepsilon}{3n}} && (1-1/x)^x \le e^{-1} \text{ for all } x\ge 1 \\
&= e^{-\tau\cdot\frac{\varepsilon}{3n}} \\
&\le e^{-\ln(2n/\delta)} \\
&= \frac{\delta}{2n},
\end{aligned}$$

as needed. □

A.3 Proof of Lemma 11

First, we prove that for all $k\ge 10n$, epoch k is ε_k-accurate with probability δ_k for $\varepsilon_k = 3n/k^2$ and $\delta_k = 2ne^{-k}$. Since $k>3n$ these are valid values between 0 and 1. Hence, we simply need to check that the τ and ℓ inequalities hold for the number of trials and number of test items specified in Algorithm 2. For arbitrary epoch k,

$$\frac{\ln(2n/\delta_k)}{\varepsilon_k/(3n)} = \ln\left(e^k\right)k^2 = k^3,$$

so the number of trials is sufficiently large. Further,

$$\begin{aligned}
\frac{18n^2}{\varepsilon_k^2}\ln\left(\frac{4k^3n}{\delta_k}\right) &= 2k^4\cdot\ln\left(2k^3e^k\right) \\
&\le 2k^4\cdot\ln\left(k^4e^k\right) && (k\ge 2) \\
&= 2k^4\cdot(k+4\ln k) \\
&\le 2k^4\cdot(k+4k) && (\ln k<k) \\
&\le 10k^5 \\
&\le k^6. && (k\ge 10)
\end{aligned}$$

Recall that $k(t)$ is defined as the epoch of item t. As in the proof of Lemma 7, we characterize deviations from the ideal algorithm in four ways.

1. Item t was allocated in one of the first $10n-1$ epochs; that is, $k(t)<10n$.
2. Item t was allocated during the sampling phase of epoch $k(t)\ge 10n$.
3. Item t was allocated during the ranking phase of epoch $k(t)\ge 10n$, which was $\varepsilon_{k(t)}$-accurate.
4. Item t was allocated during the ranking phase of epoch $k(t)\ge 10n$, which was not $\varepsilon_{k(t)}$-accurate.

We say an item t is incorrect (incorrectly allocated) when it is given to an agent with non-maximum quantile for it. We show that the number of mistakes in each category are bounded by $10^{20}n^{19}$, $2T^{5/9}$, $7nT^{17/18}$ and $1.3\cdot 10^{16}n$ respectively, with high probability. This implies, via a union bound, that the total number of mistakes is at most the sum of these quantities, or $O(\mathrm{poly}(n)\cdot T^{17/18})$, with high probability.

The number of items in category 1, is at most

$$\sum_{k=1}^{10n}k^9+k^{18}\le(10n)^{10}+(10n)^{19}\le 10^{20}n^{19}$$

Notice that $T\ge\sum_{k=1}^{k(T)-1}k^9+k^{18}\ge(k(T)-1)^{18}$, and therefore $k(T)\le 2T^{1/18}$.

For the second category, since $k(T)\le 2T^{1/18}$, the total number of items in the sampling phase is (with probability 1) upper bounded by

$$\sum_{k=1}^{k(T)}k^9\le k(T)^{10}\le 2T^{5/9}.$$

For the third category, note that each item t in this category has probability $\varepsilon_{k(t)}$ of being incorrect. The expected number of mistakes is at most

$$\sum_{k=10n}^{k(T)}\varepsilon_{k(t)}k^{18}=\sum_{k=10n}^{k(T)}3nk^{16}\le 3nk(T)^{17}\le 6nT^{17/18}.$$

Using Hoeffding's inequality we get that with high probability the number of mistakes is at most $7nT^{17/18}$, since a deviation of $nT^{17/18}$ occurs with probability at most $\exp\bigl(-2n^2T^{17/9}/T\bigr)=\exp\bigl(-2n^2T^{8/9}\bigr)$.

For the fourth category, the expected number of items in this category is at most

$$\sum_{k=10n}^{k(T)} \delta_k k^{18} = 2n \sum_{k=10}^{k(T)} \frac{k^{18}}{e^k} \leq 2n \sum_{k=1}^{\infty} \frac{k^{18}}{e^k} \leq 1.3 \cdot 10^{16} n.$$

Using Markov's inequality we have that the number of mistakes is at most $1.3 \cdot 10^{16} n \ln(T)$ with probability at least $1 - \ln(T)$, i.e., with high probability. ☐

A.4 Proof of Theorem 9

The proof is nearly identical to the proof of Theorem 5. Fix a distribution D with CDF F and let X be a random variable with distribution D. Fix some ε to be $(1-1/e) - \varepsilon$ welfare maximizing. Let $\mathcal{E}_1^T$ be the event that the maximum social welfare at time T is at least $1/2 \cdot \mathbb{E}[X] \cdot T$, let $\mathcal{E}_2^T$ be the event the ideal threshold algorithm is c-strongly-EF for $c = \frac{(\mathbb{E}[X \mid F(X) \geq 1/2] - \mathbb{E}[X])}{4n}$, let $\mathcal{E}_3^T$ be the event that the ideal threshold algorithm is a $(1-1/e)^2 - \varepsilon/2$ approximation to welfare, and let $\mathcal{E}_4^T$ be the event that Algorithm 2 differs from the ideal threshold algorithm on at most $f(T)$ items from Lemma 11. We first claim that $\mathcal{E}_1^T \cap \mathcal{E}_2^T \cap \mathcal{E}_3^T \cap \mathcal{E}_4^T$ occurs with high probability in T. Note that Lemmas 2, 1 ,and 11 tell us each of $\mathcal{E}_2^T$, $\mathcal{E}_3^T$, and $\mathcal{E}_4^T$ occur with high probability. For $\mathcal{E}_1^T$, the maximum value for each item is in expectation at least the expected value for a single agent $\mathbb{E}[X]$. Hence, a Chernoff bound tells us $\mathcal{E}_1^T$ occurs with probability at least $1 - \exp\left(\frac{-\mathbb{E}[X]T}{8}\right)$, i.e., with high probability. The claim holds because the intersection of a finite number of high probability events occurs with high probability.

Next, note that for sufficiently large T, since $f(T) \in o(T)$, $f(T) \leq \frac{(\mathbb{E}[X \mid F(X) \geq 1/2] - \mathbb{E}[X])}{8n} \cdot T$ and $f(T) \leq \varepsilon/4 \cdot \mathbb{E}[X] \cdot T$. Fix such a sufficiently large T. We show that conditioned on $\mathcal{E}_1^T \cap \mathcal{E}_2^T \cap \mathcal{E}_3^T \cap \mathcal{E}_4^T$, both EF and $((1-1/e)^2 - \varepsilon)$-welfare hold. Recall that a "difference" between Algorithm 2 and the ideal threshold algorithm refers to different distributions over the agents that get some item (i.e., a different randomized allocation). In order to make statements about envy-freeness and efficiency we need a way to argue about the differences between the algorithms ex-post. However, notice that without loss of generality we can couple the decision made by the two algorithms when randomized allocation is the same; that is, when Algorithm 2 does not differ from the ideal threshold algorithm we can assume without loss of generality that the agent who gets the item is the same. Let $A^{IT} = (A_1^{IT}, \ldots, A_n^{IT})$ be the allocation of the ideal threshold algorithm and $A = (A_1, \ldots, A_n)$ be the allocation of Algorithm 2. Beginning with envy-freeness, we have that for all pairs of agents i and j,

$$\begin{aligned} v_i(A_i) &\geq^{(\mathcal{E}_4^T)} v_i(A_i^{IT}) - f(T) \\ &\geq^{(\mathcal{E}_2^T)} v_i(A_j^{IT}) - f(T) + \frac{(\mathbb{E}[X \mid F(X) \geq 1/2] - \mathbb{E}[X])T}{4n} \\ &\geq^{(\mathcal{E}_4^T)} v_i(A_j) - 2f(T) + \frac{(\mathbb{E}[X \mid F(X) \geq 1/2] - \mathbb{E}[X])T}{4n} \\ &\geq v_i(A_j), \end{aligned}$$

so the allocation is envy-free. Let A^* be a welfare-maximizing algorithm. For the welfare approximation, we then have

$$\frac{\mathsf{sw}(A)}{\mathsf{sw}(A^*)} = \frac{\mathsf{sw}(A^{IT}) - (\mathsf{sw}(A^{IT}) - \mathsf{sw}(A))}{\mathsf{sw}(A^*)}$$

$$
\begin{aligned}
&\geq^{(\mathcal{E}_4^T)} \frac{\mathsf{sw}(A^{IT}) - f(T)}{\mathsf{sw}(A^*)} \\
&= \frac{\mathsf{sw}(A^{IT})}{\mathsf{sw}(A^*)} - \frac{f(T)}{\mathsf{sw}(A^*)} \\
&\geq^{(\mathcal{E}_3^T)} (1-1/e)^2 - \varepsilon/2 - \frac{f(T)}{\mathsf{sw}(A^*)} \\
&\geq^{(\mathcal{E}_1^T)} (1-1/e)^2 - \varepsilon/2 - \frac{f(T)}{1/2 \cdot \mathbb{E}[X] \cdot T} \\
&\geq (1-1/e)^2 - \varepsilon/2 - \frac{\varepsilon/4 \cdot \mathbb{E}[X] \cdot T}{1/2 \cdot \mathbb{E}[X] \cdot T} \\
&= (1-1/e)^2 - \varepsilon,
\end{aligned}
$$

as needed. □

A.5 Proof of Theorem 13

The proof of envy-freeness for each algorithm is nearly identical to Theorems 5 and 9 respectively; we show it here for completeness. We focus on Algorithm 1. The proof for Algorithm 2 goes through identically with all occurances of quantile maximization replaced with the ideal threshold algorithm and all occurances of Lemma 7 replaced with Lemma 11.

Fix a distributions $D_1, \ldots, D_n$ with CDFs $F_1, \ldots, F_n$ and let X_i be a random variable with distribution D_i. Fix some ε to be $(1/e - \varepsilon)$-PO. Let $E = \min_{i \in \mathcal{N}} \mathbb{E}[X_i]$ be the minimum expected value for all agents. Let $\mathcal{E}_1^T$ be the event that each agent i's value for their bundle at time T is at least $1/(2n) \cdot E \cdot T$, let $\mathcal{E}_2^T$ be the even that quantile maximization is c-strongly-EF for $c = \min_{i \in \mathcal{N}} \frac{(\mathbb{E}[X_i \mid F_i(X_i) \geq 1/2] - \mathbb{E}[X_i])}{4n}$, let $\mathcal{E}_3^T$ be the event that quantile maximization is a $(1/e - \varepsilon/2)$-PO, and let $\mathcal{E}_4^T$ be the event that Algorithm 1 differs from quantile maximization on at most $f(T)$ items from Lemma 7. We first claim that $\mathcal{E}_1^T \cap \mathcal{E}_2^T \cap \mathcal{E}_3^T \cap \mathcal{E}_4^T$ occurs with high probability in T. Note that Lemmas 1, 3, and 7 tell us $\mathcal{E}_2^T, \mathcal{E}_3^T$, and $\mathcal{E}_4^T$ each occur with high probability, respectively. For $\mathcal{E}_1^T$, note that under quantile maximization, the probability each agent i receives an item is exactly $1/n$ and the expected value conditioned on receiving the item is at least $\mathbb{E}[X_i] \geq E$. Hence, the expected contribution of each item to $v_i(A_i)$ is at least $1/n \cdot E$. A Chernoff bound then tells us $\mathcal{E}_1^T$ holds for agent i with probability at least $1 - \exp\left(\frac{-ET}{8n}\right)$. A union bound over all agent's tells us this occurs simultaneously for all agents with probability at least $1 - n\exp\left(\frac{-ET}{8n}\right)$, i.e., with high probability. The claim holds because the intersection of a finite number of high probability events occurs with high probability.

Next, note that for sufficiently large T, since $f(T) \in o(T)$, $f(T) \leq \min_{i \in \mathcal{N}} \frac{(\mathbb{E}[X_i \mid F_i(X_i) \geq 1/2] - \mathbb{E}[X_i])}{8n} \cdot T$ and $f(T) \leq \varepsilon/(4n) \cdot ET$. Fix such a sufficiently large T. We show that conditioned on $\mathcal{E}_1^T \cap \mathcal{E}_2^T \cap \mathcal{E}_3^T \cap \mathcal{E}_4^T$, both EF and $(1/e - \varepsilon)$-PO hold. Let $A^{QM} = (A_1^{QM}, \ldots, A_n^{QM})$ be the allocation of quantile maximization and $A = (A_1, \ldots, A_n)$ be the allocation of Algorithm 2. Beginning with envy-freeness, we have that for all pairs of agents i and j,

$$
\begin{aligned}
v_i(A_i) &\geq^{(\mathcal{E}_4^T)} v_i(A_i^{QM}) - f(T) \\
&\geq^{(\mathcal{E}_2^T)} v_i(A_j^{QM}) - f(T) + \min_{i \in \mathcal{N}} \frac{(\mathbb{E}[X_i \mid F_i(X_i) \geq 1/2] - \mathbb{E}[X_i])}{4n} \\
&\geq^{(\mathcal{E}_4^T)} v_i(A_j) - 2f(T) + \min_{i \in \mathcal{N}} \frac{(\mathbb{E}[X_i \mid F_i(X_i) \geq 1/2] - \mathbb{E}[X_i])}{4n}
\end{aligned}
$$

$$\geq v_i(A_j),$$

so the allocation is envy-free. Next, we show that for all agents $\frac{v_i(A_i)}{v_i(A_i^{QM})} \geq 1 - \varepsilon/2$. Since A^{QM} is $(1/e - \varepsilon/2)$-PO under $\mathcal{E}_3^T$, this implies that A is a $(1/e - \varepsilon/2)(1 - \varepsilon/2) \geq 1/e - \varepsilon$ approximation to PO as well.

To that end, for each agent i we have

$$\begin{aligned}
\frac{v_i(A_i)}{v_i(A_i^{QM})} &= \frac{v_i(A_i^{QM}) - (v_i(A_i^{QM}) - v_i(A_i))}{v_i(A_i^{QM})} \\
&= 1 - \frac{(v_i(A_i^{QM}) - v_i(A_i))}{v_i(A_i^{QM})} \\
&\geq^{(\mathcal{E}_4^T)} 1 - \frac{f(T)}{v_i(A_i^{QM})} \\
&\geq^{(\mathcal{E}_1^T)} 1 - \frac{f(T)}{ET/(2n)} \\
&\geq 1 - \frac{\varepsilon \cdot ET/(4n)}{ET/(2n)} \\
&= 1 - \varepsilon/2,
\end{aligned}$$

as needed. □